

NON-SEMISTABLE EXCEPTIONAL OBJECTS IN HEREDITARY CATEGORIES: SOME REMARKS AND CONJECTURES

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ABSTRACT. In our previous paper we studied non-semistable exceptional objects in hereditary categories and introduced the notion of regularity preserving category, but we obtained quite a few examples of such categories. Certain conditions on the Ext-nontrivial couples (exceptional objects $X, Y \in \mathcal{A}$ with $\text{Ext}^1(X, Y) \neq 0$ and $\text{Ext}^1(Y, X) \neq 0$) were shown to imply regularity-preserving. This paper is a brief review of the previous paper (with emphasis on regularity preserving property) and we add some remarks and conjectures.

It is known that in Dynkin quivers $\text{Hom}(\rho, \rho') = 0$ or $\text{Ext}^1(\rho, \rho') = 0$ for any two exceptional representations. On one hand, in the present paper we prove this fact by a new method, which allows us to extend it to representation infinite cases: the extended Dynkin quivers $\tilde{\mathbb{E}}_6, \tilde{\mathbb{E}}_7, \tilde{\mathbb{E}}_8$. On the other hand, we use it to show that for any Dynkin quiver Q there are no Ext-nontrivial couples in $\text{Rep}_k(Q)$, which implies regularity preserving of $\text{Rep}_k(Q)$, where k is an algebraically closed field.

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1. INTRODUCTION

T. Bridgeland introduced in his seminal work [1] the definition of a locally finite stability condition on a triangulated category \mathcal{T} , motivated by M. Douglas's notion of II-stability. He proved that the set of these stability conditions is a complex manifold, denoted by $\text{Stab}(\mathcal{T})$, on which act the groups $\widetilde{GL}^+(2, \mathbb{R})$ and $\text{Aut}(\mathcal{T})$. To any bounded t-structure of \mathcal{T} he assigned a set of stability conditions.

E. Macri showed in [19] that the extension closure of a full Ext-exceptional collection $\mathcal{E} = (E_0, E_1, \dots, E_n)$ in \mathcal{T} is a bounded t-structure. The stability conditions obtained from this t-structure together with their translations by the right action of $\widetilde{GL}^+(2, \mathbb{R})$ are referred to as *generated by \mathcal{E}* [19]. E. Macri, studying $\text{Stab}(D^b(K(l)))$ in [19], gave an idea for producing an exceptional pair generating a given stability condition σ on $D^b(K(l))$, where $K(l)$ is the l -Kronecker quiver.

We defined in [10] the notion of a σ -exceptional collection ([10, Definition 3.19]), so that the full σ -exceptional collections are exactly the exceptional collections which generate σ , and we focused on constructing σ -exceptional collections from a given $\sigma \in \text{Stab}(D^b(\mathcal{A}))$, where \mathcal{A} is a hereditary, hom-finite, abelian category. We developed tools for constructing σ -exceptional collections of length at least three in $D^b(\mathcal{A})$. These tools are based on the notion of *regularity-preserving hereditary category*, introduced in [10].

After a detailed study of the exceptional objects of the quiver $Q_1 = \begin{array}{ccc} & \circ & \\ \nearrow & & \nwarrow \\ \circ & \xrightarrow{\quad} & \circ \end{array}$ it was shown in [10] that $\text{Rep}_k(Q_1)$ is regularity preserving and the newly obtained methods for constructing σ -triples were applied to the case $\mathcal{A} = \text{Rep}_k(Q_1)$. As a result we obtained the following theorem:

Theorem 1.1 ([10]). *Let k be an algebraically closed field. For each $\sigma \in \text{Stab}(D^b(\text{Rep}_k(Q_1)))$ there exists a full σ -exceptional collection.*

In other words, all stability conditions on $D^b(Q_1)$ are generated by exceptional collections. This theorem implies that $\text{Stab}(D^b(Q_1))$ is connected [10, Corollary 10.2]. Theorem 1.1 and the data about the exceptional collections [10, Section 2] are a basis for working on Conjecture 7.3.

One difficulty in the proof of Theorem 1.1 is due to the Ext-nontrivial couples (Definition 2.1). In the present paper we show that this difficulty does not arise in $\text{Rep}_k(Q)$, where Q is any Dynkin quiver, which motivates us for Conjecture 7.1.

1.1. We recall now in more detail how the notion regularity preserving category appeared in [10]. By \mathcal{A} we denote a k -linear hom-finite hereditary abelian category, where k is an algebraically closed field, and we denote $D^b(\mathcal{A})$ by \mathcal{T} . We choose $\sigma \in \text{Stab}(\mathcal{T})$.

In [10, Sections 4] is given an algorithm \mathbf{alg} , which produces a distinguished triangle $\mathbf{alg}(R) = U \rightarrow R \rightarrow V \rightarrow U[1]$ with semi-stable V , from any unstable exceptional object $R \in \mathcal{T}$. The triangle $\mathbf{alg}(R)$ satisfies the vanishings $\text{hom}^1(U, U) = \text{hom}^1(V, V) = \text{hom}^*(U, V) = 0$ for any unstable exceptional object E , provided that the category \mathcal{A} has no Ext-nontrivial couples. When \mathcal{A} contains Ext-nontrivial couples, these vanishings are not guaranteed by \mathbf{alg} . By definition we call a non-semistable exceptional object R σ -regular, when these vanishings hold for $\mathbf{alg}(R)$, otherwise we call it σ -irregular. In particular, if \mathcal{A} has no Ext-nontrivial couples, then all non-semistable objects are σ -regular. When R is σ -regular, the vanishings in $\mathbf{alg}(R)$ imply that for any indecomposable components S and E of V and U , respectively, the pair (S, E) is exceptional with semistable first element S . We denote this relation between a σ -regular object R and the exceptional pair (S, E) by $R \xrightarrow{X} (S, E)$, where X contains further information (see [10, Definition 5.2]).

The σ -regular objects in turn are divided into final and non-final as follows. In each relation $R \xrightarrow{X} (S, E)$ the first component S is a semistable exceptional object, and the second is not restricted to be always semistable. If there is such a relation with a non-semistable E , then we refer to R as a *nonfinal* σ -regular object, otherwise - *final*. The name nonfinal is justified, when the category \mathcal{A} has a specific property called regularity-preserving, defined as follows:

Definition 1.2. ([10, Definition 6.1]) *A hereditary abelian category \mathcal{A} will be said to be regularity-preserving, if for each $\sigma \in \text{Stab}(D^b(\mathcal{A}))$ from the the following data:*

$R \in D^b(\mathcal{A})$ is a σ -regular object; $R \xrightarrow{X} (S, E)$; $E \notin \sigma^{ss}$

it follows that E is a σ -regular object as well.

In a regularity-preserving category \mathcal{A} the relation $\cdots \dashrightarrow$ circumvents the σ -irregular objects, and each non-final σ -regular object R generates a long sequence¹ of the form:

$$(1) \quad \begin{array}{ccccccc} R & \dashrightarrow^{X_1} & (S_1, E_1) & \xrightarrow{\text{proj}_2} & E_1 & \dashrightarrow^{X_2} & (S_2, E_2) & \xrightarrow{\text{proj}_2} & E_2 & \dashrightarrow^{X_3} & (S_3, E_3) & \xrightarrow{\text{proj}_2} & \dots \\ & & \text{proj}_1 \downarrow & & & & \text{proj}_1 \downarrow & & & & \text{proj}_1 \downarrow & & \\ & & S_1 & & & & S_2 & & & & S_3 & & \end{array}.$$

Furthermore, after finitely many steps (say n) a final σ -regular object E_n appears ([10, Lemma 7.1]) and then $S_1, S_2, \dots, S_{n+1}, E_{n+1}$ ($n \geq 1$) is a sequence of semistable and exceptional objects. The last pair (S_{n+1}, E_{n+1}) is always exceptional, however the entire sequence is not always an exceptional collection. The sequences of the form (1) generated by σ -regular objects are the main tool used in [10, Sections 7,8,9] for constructing σ -exceptional collections.

1.2. We explain now the known examples of regularity preserving categories. Apart from its expository aspect, this paper enlarges the small list of examples of such categories.

In [10, Section 6] we found certain conditions on the Ext-nontrivial couples of \mathcal{A} , called *RP property 1* and *RP property 2* (see Definition 2.2 below), which imply regularity-preserving. In particular, non-existing of such couples implies this property. Thus, regularity preserving property is related to specific pairwise relations between the exceptional objects of \mathcal{A} .

The study of exceptional objects in quivers goes back to [22], [23], [8], and to [20] for more general hereditary categories. However, to the best of our knowledge, no attention to the Ext-nontrivial couples has been focused.

It follows from [19, Lemma 4.1] that there are no Ext-nontrivial couples in $\text{Rep}_k(K(l))$ and hence $\text{Rep}_k(K(l))$ is an example of regularity preserving category (see [10, Appendix B]).

Two more examples are the categories $\text{Rep}_k(Q_1)$ and $\text{Rep}_k(Q_2)$ (the quiver Q_2 is in Fig. (2) in Section 2). In [10] is shown that there are Ext-nontrivial couples in $\text{Rep}_k(Q_1)$ and $\text{Rep}_k(Q_2)$, but after computing the dimensions of $\text{Hom}(X, Y)$, $\text{Ext}^1(X, Y)$ for any two exceptional objects X, Y and by a careful analysis of the results we showed that RP properties 1 and 2 do hold in both the cases. Furthermore, the resulting tables of dimensions show that one of the spaces $\text{Hom}(X, Y)$, $\text{Ext}^1(X, Y)$ always vanishes.

The new (and easy) examples given in the present paper are the categories of representations of Dynkin quivers (Corollary 6.5, Section 6).

1.3 We describe now the content of the present paper.

The basic observation is that if a quiver Q satisfies $\text{Hom}_Q(\rho, \rho') = 0$ or $\text{Ext}_Q^1(\rho, \rho') = 0$ for any two exceptional representations ρ, ρ' , then the dimension vectors of any Ext-nontrivial couple $\{\rho, \rho'\}$ satisfy $\langle \underline{\dim}(\rho) + \underline{\dim}(\rho'), \underline{\dim}(\rho) + \underline{\dim}(\rho') \rangle \leq 0$ (Lemma 3.1). This motivates us to study in more detail the property that $\text{Hom}(\rho, \rho') = 0$ or $\text{Ext}^1(\rho, \rho') = 0$ for given exceptional representations $\rho, \rho' \in \text{Rep}_k(Q)$. In Section 2 we recall some results from [10] about the exceptional objects of $\text{Rep}_k(Q_1)$, $\text{Rep}_k(Q_2)$, in particular this property holds in $\text{Rep}_k(Q_1)$ and $\text{Rep}_k(Q_2)$ for any two exceptional representations (Corollary 2.5 (b)). An example with an acyclic quiver where this fails is obtained by changing the orientation of the quiver Q_2 (see (17)). In particular, the category of representations changes by changing the orientation of the arrows and keeping the quiver acyclic (Lemma 3.7).

¹By “long” we mean that it has at least two steps. This sequence is not uniquely determined by R .

In Section 3 is recalled the definition of the standard differential in the 2-term complex computing $\mathbb{R}\mathrm{Hom}_Q(\rho, \rho')$ for any two representations $\rho, \rho' \in \mathrm{Rep}_k(Q)$, which we denote by $F_{\rho, \rho'}^Q$. We utilize this linear map because the condition that one of the two spaces $\mathrm{Hom}(\rho, \rho')$ or $\mathrm{Ext}^1(\rho, \rho')$ vanishes is the same as the condition that $F_{\rho, \rho'}^Q$ has maximal rank. In Sections 4, 5 we find conditions which ensure maximality of the rank of $F_{\rho, \rho'}^Q$. The strategy is to expand the simple linear-algebraic observations: Lemma 5.1 (a),(b) and Lemma 4.5 to big enough quivers by using Corollary 3.5.² The obtained conditions, which ensure maximality of the rank, are as follows.

Let ρ, ρ' be exceptional representations, α, α' be their dimension vectors and let A, A' be the supports of α, α' . When Q has no edges loops and α or α' has only one nontrivial value, i. e. A or A' is a single element set, then $F_{\rho, \rho'}^Q$ has maximal rank (Lemma 3.6).

In Section 4 we consider quivers without loops and exceptional representations whose dimension vectors are thin, i. e. the components of these vectors take values in $\{0, 1\}$ (see Definition 4.2). The main result of this section (Lemma 4.4) is that, when the graph of Q has no loops, for any two thin exceptional representations ρ, ρ' the linear map $F_{\rho, \rho'}^Q$ has maximal rank. The last Lemma 4.6 of this section considers some cases in which $A \cap A'$ is a single element set and ρ, ρ' are not restricted to be with thin dimension vectors.

In Section 5 we restrict Q further. We consider star shaped quivers with any orientation of the arrows (see Figure 29). We allow here the exceptional representations to have hill dimension vector (Definition 5.2) in addition to thin dimension vectors. It is shown that for any two exceptional representations $\rho, \rho' \in \mathrm{Rep}_k(Q)$, whose dimension vectors are hill or thin, the map $F_{\rho, \rho'}^Q$ has maximal rank (Proposition 5.6).

A natural question is whether the dimension vectors of all exceptional representations in a given star shaped quiver are either hill or thin ?

Since the answer of this question is positive for any Dynkin quiver Q (Remark 6.1) and for the extended Dynkin quivers $\tilde{\mathbb{E}}_6, \tilde{\mathbb{E}}_7, \tilde{\mathbb{E}}_8$ (Lemma 6.2), it follows that for any two exceptional representations ρ, ρ' in a Dynkin quiver Q or in an extended Dynkin quiver of the type $\tilde{\mathbb{E}}_6, \tilde{\mathbb{E}}_7, \tilde{\mathbb{E}}_8$ the linear map $F_{\rho, \rho'}^Q$ has maximal rank (Corollary 6.3). Thus, in these quivers we have $\mathrm{Hom}(\rho, \rho') = 0$ or $\mathrm{Ext}^1(\rho, \rho') = 0$. This property for Dynkin quivers follows easily from the fact that $\mathrm{Rep}_k(Q)$ is representation directed for Dynkin Q (see [21, p. 59] for the argument and [4], [12] for the fact that Dynkin quivers are representation directed).³ However $\tilde{\mathbb{E}}_6, \tilde{\mathbb{E}}_7, \tilde{\mathbb{E}}_8$ are not representation directed, since they are representation-infinite (see [24, 5.5, p. 307]), so this argument can not be applied to them. We expect that the arguments in Sections 3, 4, 5 can be extended further to show that $\mathrm{Hom}(\rho, \rho') = 0$ or $\mathrm{Ext}^1(\rho, \rho') = 0$ for any two exceptional representations ρ, ρ' in $\tilde{\mathbb{D}}_n$ for $n \geq 5$.

Star-shaped quivers have been extensively studied (going back to [14], [4] and recently e.g. [15]), but to the best of our knowledge Proposition 5.6 is new.

From here till the end of the introduction Q is a Dynkin quiver Q (i. e. the graph of Q is A_n with $n \geq 1$ or D_n with $n \geq 4$ or E_n with $n = 6, 7, 8$). Lemma 3.1 combined with Corollary 6.3 and the positivity of the Euler form imply that there are no Ext-nontrivial couples in $\mathrm{Rep}_k(Q)$

²Corollary 3.5 is based on the algebro-geometric fact (see e.g. [9, p. 13]) that the orbit \mathcal{O}_ρ of an exceptional representation ρ is Zariski open in a certain affine space.

³We thank Pranav Pandit for pointing us these references.

(Corollary 6.4). This in turn implies that $\text{Rep}_k(Q)$ is regularity preserving. Furthermore, there are no σ -irregular objects for any $\sigma \in \text{Stab}(D^b(\text{Rep}_k(Q)))$.

Corollary 6.4 means that for any two exceptional representations $\rho, \rho' \in \text{Rep}_k(Q)$ we have $\text{Ext}^1(\rho, \rho') = 0$ or $\text{Ext}^1(\rho', \rho) = 0$. Analogous property in degree zero, which says that $\text{Hom}(\rho, \rho') = 0$ or $\text{Hom}(\rho', \rho) = 0$ for any two non-equivalent exceptional representations $\rho, \rho' \in \text{Rep}_k(Q)$, is well known for Dynkin quivers. These facts about any Dynkin quiver Q can be summarized by saying that for any two non-equivalent exceptional representations $\rho, \rho' \in \text{Rep}_k(Q)$ the product of the two numbers in each row and in each column of the table below vanishes (see **Some notations** for the notations $\text{hom}(\rho, \rho')$, $\text{hom}^1(\rho, \rho')$):

$\text{hom}(\rho, \rho')$	$\text{hom}^1(\rho, \rho')$
$\text{hom}(\rho', \rho)$	$\text{hom}^1(\rho', \rho)$

In the final Section 7 we pose some conjectures and questions.

Some notations. In these notes k is an algebraically closed field. The letter \mathcal{T} denotes a triangulated category, linear over k , the shift functor in \mathcal{T} is designated by $[1]$. We write $\text{Hom}^i(X, Y)$ for $\text{Hom}(X, Y[i])$ and $\text{hom}^i(X, Y)$ for $\dim_k(\text{Hom}(X, Y[i]))$, where $X, Y \in \mathcal{T}$.

An *exceptional object* is an object $E \in \mathcal{T}$ satisfying $\text{Hom}^i(E, E) = 0$ for $i \neq 0$ and $\text{Hom}(E, E) = k$. We denote by \mathcal{A}_{exc} , resp. $D^b(\mathcal{A})_{exc}$, the set of all exceptional objects of \mathcal{A} , resp. of $D^b(\mathcal{A})$.

An *exceptional collection* is a sequence $\mathcal{E} = (E_1, E_2, \dots, E_n) \subset \mathcal{T}_{exc}$ satisfying $\text{hom}^*(E_i, E_j) = 0$ for $i > j$. If in addition we have $\langle \mathcal{E} \rangle = \mathcal{T}$, then \mathcal{E} will be called a full exceptional collection.

An abelian category \mathcal{A} is said to be hereditary, if $\text{Ext}^i(X, Y) = 0$ for any $X, Y \in \mathcal{A}$ and $i \geq 2$, it is said to be of finite length, if it is Artinian and Noetherian.

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2. THE TWO EXAMPLES IN [10] OF REGULARITY PRESERVING CATEGORIES WITH EXT-NONTRIVIAL COUPLES

Here we recall some results and definitions of [10, Section 2], which are related to the results of the present paper. First we recall two definitions

Definition 2.1. An *Ext-nontrivial couple* in a hereditary abelian category \mathcal{A} is a couple of exceptional objects $\{L, \Gamma\} \subset \mathcal{A}_{exc}$, s. t. $\text{hom}^1(L, \Gamma) \neq 0$ and $\text{hom}^1(\Gamma, L) \neq 0$.

Definition 2.2. Let \mathcal{A} be a hereditary category. We say that \mathcal{A} has

RP Property 1: if for each Ext-nontrivial couple $\{\Gamma, \Gamma'\} \subset \mathcal{A}$ and for each $X \in \mathcal{A}_{exc}$ from $\text{hom}^*(\Gamma, X) = 0$ it follows $\text{hom}^*(X, \Gamma') = 0$;

RP Property 2: if for each Ext-nontrivial couple $\{\Gamma, \Gamma'\} \subset \mathcal{A}$ and for any two $X, Y \in \mathcal{A}_{exc}$ from $\text{hom}(\Gamma, X) \neq 0$, $\text{hom}(X, Y) \neq 0$, $\text{hom}^*(\Gamma, Y) = 0$ it follows $\text{hom}(\Gamma', Y) \neq 0$.

For any finite quiver Q we denote the category of k -representations of Q by $\text{Rep}_k(Q)$. This category is a hom-finite hereditary k -linear abelian (see e. g. [9]). In this section are discussed the exceptional objects and their pairwise relations in $\text{Rep}_k(Q_1)$, $\text{Rep}_k(Q_2)$, where:

$$(2) \quad Q_1 = \begin{array}{ccc} & \circ & \\ \nearrow & & \nwarrow \\ \circ & \xrightarrow{\quad} & \circ \end{array} \quad Q_2 = \begin{array}{ccc} \circ & \xrightarrow{\quad} & \circ \\ \uparrow & & \uparrow \\ \circ & \xrightarrow{\quad} & \circ \end{array}.$$

Recall (see page 8 in [9]) that for any quiver Q and any $\rho, \rho' \in \text{Rep}_k(Q)$ we have the formula

$$(3) \quad \text{hom}(\rho, \rho') - \text{hom}^1(\rho, \rho') = \langle \underline{\dim}(\rho), \underline{\dim}(\rho') \rangle,$$

where \langle, \rangle is the Euler form of Q . In particular, it follows that if $\rho \in \text{Rep}_k(Q)$ is an exceptional object, then $\langle \underline{\dim}(\rho), \underline{\dim}(\rho) \rangle = 1$. The vectors satisfying this equality are called real roots (see [9, p. 17]). The real roots of Q_1 are $(m+1, m, m), (m, m+1, m+1), (m, m, m+1), (m+1, m+1, m), (m+1, m, m+1), (m, m+1, m)$, $m \geq 0$. The imaginary roots⁴ of Q_1 , are (m, m, m) , $m \geq 1$. Not every real root is a dimension vector of an exceptional representation (see [10, Lemma 2.1]).

Propositions 2.3 and 2.4 classify the exceptional objects on $\text{Rep}_k(Q_1)$, $\text{Rep}_k(Q_2)$.

Proposition 2.3 ([10]). *The exceptional objects up to isomorphism in $\text{Rep}_k(Q_1)$ are $(m = 0, 1, 2, \dots)$*

$$\begin{array}{lll} E_1^m = \begin{array}{ccc} & k^m & \\ \pi_+^m \nearrow & & \nwarrow \text{Id} \\ k^{m+1} & \xrightarrow{\pi_-^m} & k^m \end{array} & E_2^m = \begin{array}{ccc} & k^{m+1} & \\ j_+^m \nearrow & & \nwarrow \text{Id} \\ k^m & \xrightarrow{j_-^m} & k^{m+1} \end{array} & E_3^m = \begin{array}{ccc} & k^{m+1} & \\ j_+^m \nearrow & & \nwarrow j_-^m \\ k^m & \xrightarrow{\text{Id}} & k^m \end{array} \\ E_4^m = \begin{array}{ccc} & k^m & \\ \pi_+^m \nearrow & & \nwarrow \pi_-^m \\ k^{m+1} & \xrightarrow{\text{Id}} & k^{m+1} \end{array} & M = \begin{array}{ccc} & 0 & \\ \nearrow & & \nwarrow \\ 0 & \xrightarrow{\quad} & k \end{array} & M' = \begin{array}{ccc} & k & \\ \text{Id} \nearrow & & \nwarrow \\ k & \xrightarrow{\quad} & 0 \end{array} \end{array}$$

where

$$\begin{aligned} \pi_+^m(a_1, a_2, \dots, a_m, a_{m+1}) &= (a_1, a_2, \dots, a_m) & \pi_-^m(a_1, a_2, \dots, a_m, a_{m+1}) &= (a_2, \dots, a_m, a_{m+1}) \\ j_+^m(a_1, a_2, \dots, a_m) &= (a_1, a_2, \dots, a_m, 0) & j_-^m(a_1, a_2, \dots, a_m) &= (0, a_1, \dots, a_m). \end{aligned}$$

Proposition 2.4 ([10]). *The exceptional objects up to isomorphism in $\text{Rep}_k(Q_2)$ are $(m = 0, 1, 2, \dots)$*

$$\begin{array}{llll} E_1^m = \begin{array}{ccc} k^m & \xrightarrow{\text{Id}} & k^m \\ \pi_+^m \uparrow & & \uparrow \text{Id} \\ k^{m+1} & \xrightarrow{\pi_-^m} & k^m \end{array} & E_2^m = \begin{array}{ccc} k^{m+1} & \xrightarrow{\text{Id}} & k^{m+1} \\ j_+^m \uparrow & & \uparrow \text{Id} \\ k^m & \xrightarrow{j_-^m} & k^{m+1} \end{array} & E_3^m = \begin{array}{ccc} k^m & \xrightarrow{j_+^m} & k^{m+1} \\ \text{Id} \uparrow & & \uparrow j_-^m \\ k^m & \xrightarrow{\text{Id}} & k^m \end{array} & E_4^m = \begin{array}{ccc} k^{m+1} & \xrightarrow{\pi_+^m} & k^m \\ \text{Id} \uparrow & & \uparrow \pi_-^m \\ k^{m+1} & \xrightarrow{\text{Id}} & k^{m+1} \end{array} \\ E_5^m = \begin{array}{ccc} k^m & \xrightarrow{j_+^m} & k^{m+1} \\ \text{Id} \uparrow & & \uparrow \text{Id} \\ k^m & \xrightarrow{j_-^m} & k^{m+1} \end{array} & E_6^m = \begin{array}{ccc} k^{m+1} & \xrightarrow{\pi_+^m} & k^m \\ \text{Id} \uparrow & & \uparrow \text{Id} \\ k^{m+1} & \xrightarrow{\pi_-^m} & k^m \end{array} & E_7^m = \begin{array}{ccc} k^m & \xrightarrow{\text{Id}} & k^m \\ \pi_+^m \uparrow & & \uparrow \pi_-^m \\ k^{m+1} & \xrightarrow{\text{Id}} & k^{m+1} \end{array} & E_8^m = \begin{array}{ccc} k^{m+1} & \xrightarrow{\text{Id}} & k^{m+1} \\ j_+^m \uparrow & & \uparrow j_-^m \\ k^m & \xrightarrow{\text{Id}} & k^m \end{array} \\ F_+ = \begin{array}{ccc} k & \rightarrow & 0 \\ \uparrow & & \uparrow \\ 0 & \rightarrow & 0 \end{array} & F_- = \begin{array}{ccc} 0 & \rightarrow & 0 \\ \uparrow & & \uparrow \\ 0 & \rightarrow & k \end{array} & G_+ = \begin{array}{ccc} k & \xrightarrow{\text{Id}} & k \\ \text{Id} \uparrow & & \uparrow \\ k & \rightarrow & 0 \end{array} & G_- = \begin{array}{ccc} 0 & \rightarrow & k \\ \uparrow & & \uparrow \\ k & \xrightarrow{\text{Id}} & k \end{array} \end{array}$$

In [10, Subsection 2.2] we compute $\text{hom}(\rho, \rho')$, $\text{hom}^1(\rho, \rho')$ with ρ, ρ' varying throughout the obtained lists in Propositions 2.3, 2.4 and group these dimensions in tables of series. From these tables one finds that the only Ext-nontrivial couple in $\text{Rep}_k(Q_1)$ is $\{M, M'\}$ and the Ext-nontrivial couples in $\text{Rep}_k(Q_2)$ are $\{F_+, G_-\}$, $\{F_-, G_+\}$.

From the obtained tables with dimensions one verifies also the following properties.

⁴Imaginary root is a vector ρ with $\langle \underline{\dim}(\rho), \underline{\dim}(\rho) \rangle \leq 0$.

Corollary 2.5 ([10]). *The categories $\text{Rep}_k(Q_1)$, $\text{Rep}_k(Q_2)$ satisfy the following properties:*

- (a) *RP property 1, RP property 2 (see Definition 2.2).*
- (b) *For any two exceptional objects $X, Y \in \text{Rep}_k(Q_i)$ at most one degree in $\{\text{hom}^p(X, Y)\}_{p \in \mathbb{Z}}$ is nonzero, where $i \in \{1, 2\}$.*

3. THE DIFFERENTIAL $F_{\rho, \rho'}^Q$

In this section Q is any connected quiver and \langle, \rangle is the Euler form of Q . We denote the set of vertices by $V(Q)$, the set of arrows by $\text{Arr}(Q)$, and the underlying non-oriented graph by $\Gamma(Q)$. Let

$$(4) \quad \text{Arr}(Q) \rightarrow V(Q) \times V(Q) \quad a \mapsto (s(a), t(a)) \in V(Q) \times V(Q)$$

be the function assigning to an arrow $a \in \text{Arr}(Q)$ its origin $s(a) \in V(Q)$ and its end $t(a) \in V(Q)$. The dual quiver Q^\vee has $V(Q^\vee) = V(Q)$, $\text{Arr}(Q^\vee) = \text{Arr}(Q)$, but $(s^\vee, t^\vee) = (t, s)$. By transposing matrices we obtain an equivalence

$$(5) \quad \text{Rep}_k(Q)^{op} \xrightarrow{\vee} \text{Rep}_k(Q^\vee).$$

The following properties hold

$$(6) \quad \forall \rho, \rho' \in \text{Rep}_k(Q) \quad \underline{\dim}(\rho) = \underline{\dim}(\rho^\vee) \quad \text{hom}_Q^i(\rho, \rho') = \text{hom}_{Q^\vee}^i(\rho'^\vee, \rho^\vee)$$

$$(7) \quad \forall \alpha, \beta \in \mathbb{N}^{V(Q)} \quad \langle \alpha, \beta \rangle_{Q^\vee} = \langle \beta, \alpha \rangle_Q$$

The basic observation of this paper is:

Lemma 3.1. *If any two exceptional objects $\rho, \rho' \in \text{Rep}_k(Q)$ satisfy $\text{hom}(\rho, \rho') = 0$ or $\text{hom}^1(\rho, \rho') = 0$, then any Ext-nontrivial couple $\{\rho, \rho'\}$ satisfies $\langle \underline{\dim}(\rho) + \underline{\dim}(\rho'), \underline{\dim}(\rho) + \underline{\dim}(\rho') \rangle \leq 0$.*

Proof. Since ρ, ρ' are exceptional representations, we have $\langle \underline{\dim}(\rho), \underline{\dim}(\rho) \rangle = \langle \underline{\dim}(\rho'), \underline{\dim}(\rho') \rangle = 1$, therefore

$$(8) \quad \langle \underline{\dim}(\rho) + \underline{\dim}(\rho'), \underline{\dim}(\rho) + \underline{\dim}(\rho') \rangle = 2 + \langle \underline{\dim}(\rho), \underline{\dim}(\rho') \rangle + \langle \underline{\dim}(\rho'), \underline{\dim}(\rho) \rangle.$$

Since $\text{hom}^1(\rho, \rho') \neq 0$, $\text{hom}^1(\rho', \rho) \neq 0$, by the given property of the exceptional objects we obtain $\text{hom}(\rho, \rho') = \text{hom}(\rho', \rho) = 0$, hence by (3) we obtain $\langle \underline{\dim}(\rho), \underline{\dim}(\rho') \rangle = -\text{hom}^1(\rho, \rho') < 0$, $\langle \underline{\dim}(\rho'), \underline{\dim}(\rho) \rangle = -\text{hom}^1(\rho', \rho) < 0$. Now the lemma follows from (8). \square

In Corollary 2.5 (b) we see that the condition of the lemma above is satisfied in $\text{Rep}_k(Q_1)$, $\text{Rep}_k(Q_2)$. The condition of Lemma 3.1 is related to the standard differential in the 2-term complex computing $\mathbb{R}\text{Hom}_Q(\rho, \rho')$. We recall this definition:

Definition 3.2. *For any two representations $\rho, \rho' \in \text{Rep}_k(Q)$ we denote by $F_{\rho, \rho'}^Q$ (we omit the superscript Q , when it is clear which is the quiver in question) the standard differential in the 2-term complex computing $\mathbb{R}\text{Hom}_Q(\rho, \rho')$. Recall that:*

$$(9) \quad F_{\rho, \rho'}^Q : \prod_{i \in V(Q)} \text{Hom}(k^{\alpha_i}, k^{\alpha'_i}) \rightarrow \prod_{a \in \text{Arr}(Q)} \text{Hom}(k^{\alpha_{s(a)}}, k^{\alpha'_{t(a)}})$$

where $\alpha = \underline{\dim}(\rho)$, $\alpha' = \underline{\dim}(\rho') \in \mathbb{N}^{V(Q)}$, as follows:

$$(10) \quad F_{\rho, \rho'}^Q(\{f_i\}_{i \in V(Q)}) = \{f_{t(a)} \circ \rho_a - \rho'_a \circ f_{s(a)}\}_{a \in \text{Arr}(Q)}.$$

This differential will be used to obtain the condition in Lemma 3.1. More precisely, we have the following standard facts:

Lemma 3.3. *Let Q be a quiver and $\rho, \rho' \in \text{Rep}_k(Q)$ be two representations. The following hold:*

- (a) $\text{Hom}_{\text{Rep}_k(Q)}(\rho, \rho') = \ker(F_{\rho, \rho'}^Q)$
- (b) $\langle \underline{\dim}(\rho), \underline{\dim}(\rho') \rangle = \dim(\text{dom}(F_{\rho, \rho'}^Q)) - \dim(\text{cod}(F_{\rho, \rho'}^Q))$, where $\text{dom}(F_{\rho, \rho'}^Q)$ and $\text{cod}(F_{\rho, \rho'}^Q)$ denote the domain and codomain of $F_{\rho, \rho'}^Q$.
- (c) Let $\langle \underline{\dim}(\rho), \underline{\dim}(\rho') \rangle \geq 0$. Then $F_{\rho, \rho'}^Q$ has maximal rank iff $\text{hom}(\rho, \rho') = \langle \underline{\dim}(\rho), \underline{\dim}(\rho') \rangle$ and $\text{hom}^1(\rho, \rho') = 0$.
- (d) Let $\langle \underline{\dim}(\rho), \underline{\dim}(\rho') \rangle < 0$. Then $F_{\rho, \rho'}^Q$ has maximal rank iff $\text{hom}(\rho, \rho') = 0$ and $\text{hom}^1(\rho, \rho') = -\langle \underline{\dim}(\rho), \underline{\dim}(\rho') \rangle$.
- (e) Let Q^\vee be the dual quiver and \vee be the equivalence in (5), then $F_{\rho, \rho'}^Q$ has maximal rank iff $F_{\rho^\vee, \rho'^\vee}^{Q^\vee}$ has maximal rank

Proof. (a) and (b) follow from the definitions, (c) and (d) follow from (a), (b) and (3). Finally, (e) follows from (c), (d), (6), and (7). \square

For any $\alpha \in \mathbb{N}^{V(Q)}$ we denote

$$GL(\alpha) = \prod_{i \in V(Q)} GL(\alpha_i, k); \text{Rep}(\alpha) = \{\rho \in \text{Rep}_k(Q) : \underline{\dim}(\rho) = \alpha\} = \prod_{a \in \text{Arr}(Q)} \text{Hom}(k^{\alpha_{s(a)}}, k^{\alpha_{t(a)}}).$$

For any $\alpha \in \mathbb{N}^{V(Q)}$ the isomorphism classes of representations with dimension vector α are the orbits of the left action:

$$(11) \quad GL(\alpha) \times \text{Rep}(\alpha) \rightarrow \text{Rep}(\alpha) \quad g \cdot \rho = \{g(t(a)) \circ \rho_a \circ g(s(a))^{-1}\}_{a \in \text{Arr}(Q)}.$$

For $\rho \in \text{Rep}(\alpha)$ the orbit containing ρ is denoted by \mathcal{O}_ρ .

Let $\alpha, \alpha' \in \mathbb{N}^{V(Q)}$, $g \in GL(\alpha)$, $g' \in GL(\alpha')$. It is easy to show that for any $\rho \in \text{Rep}(\alpha)$, $\rho' \in \text{Rep}(\alpha')$ we have

$$(12) \quad F_{g \cdot \rho, \rho'} = R_{g^{-1}} \circ F_{\rho, \rho'} \circ R_g \quad F_{\rho, g' \cdot \rho'} = L_{g'} \circ F_{\rho, \rho'} \circ L_{g'^{-1}},$$

where:

$$(13) \quad L_{g'}, R_g : \prod_{i \in V(Q)} \text{Hom}(k^{\alpha_i}, k^{\alpha'_i}) \rightarrow \prod_{i \in V(Q)} \text{Hom}(k^{\alpha_i}, k^{\alpha'_i})$$

$$L_{g'}(\{f_i\}_{i \in V(Q)}) = \{g'_i \circ f_i\}_{i \in V(Q)}, \quad R_g(\{f_i\}_{i \in V(Q)}) = \{f_i \circ g_i\}_{i \in V(Q)};$$

$$(14) \quad L_{g'}, R_g : \prod_{a \in \text{Arr}(Q)} \text{Hom}(k^{\alpha_{s(a)}}, k^{\alpha'_{t(a)}}) \rightarrow \prod_{a \in \text{Arr}(Q)} \text{Hom}(k^{\alpha_{s(a)}}, k^{\alpha'_{t(a)}})$$

$$L_{g'}(\{u_a\}_{a \in \text{Arr}(Q)}) = \{g'_{t(a)} \circ u_a\}_{a \in \text{Arr}(Q)} \quad R_g(\{u_a\}_{a \in \text{Arr}(Q)}) = \{u_a \circ g_{s(a)}\}_{a \in \text{Arr}(Q)}.$$

In particular, we see immediately that

Lemma 3.4. *Let $\alpha, \alpha' \in \mathbb{N}^{V(Q)}$, $(\rho, \rho') \in \text{Rep}(\alpha) \times \text{Rep}(\alpha')$. If $F_{\rho, \rho'}$ is not of maximal rank, then $F_{x, y}$ is not of maximal rank for any $(x, y) \in \mathcal{O}_\rho \times \mathcal{O}_{\rho'}$.*

The following corollary will be widely used.

Corollary 3.5. *Let $\alpha, \alpha' \in \mathbb{N}^{V(Q)}$ be real roots of Q . Let $\rho \in \text{Rep}(\alpha)$, $\rho' \in \text{Rep}(\alpha')$ be exceptional representations. If $F_{x, y}$ has maximal rank for some $(x, y) \in \text{Rep}(\alpha) \times \text{Rep}(\alpha')$, then $F_{\rho, y}$ and $F_{x, \rho'}$ have maximal rank. For each $a \in \text{Arr}(Q)$ the linear maps ρ_a, ρ'_a have maximal rank.*

Proof. First recall that, since ρ, ρ' are exceptional, the orbits \mathcal{O}_ρ and $\mathcal{O}_{\rho'}$ are Zariski open in $\text{Rep}(\alpha)$ and $\text{Rep}(\alpha')$, respectively (see [9, p. 13]). For a given $x \in \text{Rep}(\alpha)$ the condition on $y \in \text{Rep}(\alpha')$ to be such that $F_{x, y}$ is not of maximal rank is expressed by vanishing of certain family of polynomials on $\text{Rep}(\alpha')$. If there is $y \in \text{Rep}(\alpha')$ such that $F_{x, y}$ is of maximal rank, then the zero set of this family of polynomials is a proper Zariski closed subset of $\text{Rep}(\alpha')$, hence, by the previous lemma, not maximality of the rank of $F_{x, \rho'}$ implies that the orbit $\mathcal{O}_{\rho'}$ is contained in this proper Zariski closed subset, and then $\mathcal{O}_{\rho'}$ can not be an open subset of $\text{Rep}(\alpha')$. Thus, we showed that if $F_{x, y}$ is of maximal rank for some $y \in \text{Rep}(\alpha')$, then $F_{x, \rho'}$ is of maximal rank. The claim about $F_{\rho, y}$ is proved by the same arguments applied to ρ .

Finally, the property that ρ_a is not of maximal rank is invariant under the action of $GL(\alpha)$, for any $a \in \text{Arr}(Q)$. It follows that non-maximality of the rank of ρ_a implies that \mathcal{O}_ρ is contained in a proper Zariski closed subset of $\text{Rep}(\alpha)$. If ρ is an exceptional representation, then \mathcal{O}_ρ is Zariski open in $\text{Rep}(\alpha)$, therefore ρ_a is of maximal rank for each $a \in \text{Arr}(Q)$. \square

It is useful to give a more precise description of the map defined in Definition 3.2. For any $(\rho, \rho') \in \text{Rep}(\alpha) \times \text{Rep}(\alpha')$ we denote $A = \{i \in V(Q) : \alpha_i \neq 0\}$, $A' = \{i \in V(Q) : \alpha'_i \neq 0\}$. We denote also $\text{Arr}(A, A') = \{a \in \text{Arr}(Q) : s(a) \in A, t(a) \in A'\}$. Then for $F_{\rho, \rho'}$ we can write

$$(15) \quad F_{\rho, \rho'} : \prod_{i \in A \cap A'} \text{Hom}(k^{\alpha_i}, k^{\alpha'_i}) \rightarrow \prod_{a \in \text{Arr}(A, A')} \text{Hom}(k^{\alpha_{s(a)}}, k^{\alpha'_{t(a)}})$$

$$(16) \quad F_{\rho, \rho'}(\{f_i\}_{i \in A \cap A'}) = \left\{ \begin{array}{ll} f_{t(a)} \circ \rho_a - \rho'_a \circ f_{s(a)} & a \in \text{Arr}(A \cap A', A \cap A') \\ -\rho'_a \circ f_{s(a)} & a \in \text{Arr}(A \cap A', A' \setminus A) \\ f_{t(a)} \circ \rho_a & a \in \text{Arr}(A \setminus A', A \cap A') \\ 0 & a \in \text{Arr}(A \setminus A', A' \setminus A) \end{array} \right\}.$$

In the rest sections we will study the question about maximality of the rank of $F_{\rho, \rho'}$, where ρ, ρ' are exceptional representations. We prove first the following lemma:

Lemma 3.6. *Let Q have no edges loops. Let $\rho, \rho' \in \text{Rep}_k(Q)$ be exceptional representations. If A or A' is a single element set, then $F_{\rho, \rho'}$ has maximal rank.*

Proof. Recall that we denote $\alpha = \underline{\dim}(\rho)$, $\alpha' = \underline{\dim}(\rho')$.

Assume that $A = \{j\}$. If $A \cap A' = \emptyset$, then $F_{\rho, \rho'}$ is injective. Let $A \cap A' = \{j\}$.

Now $A \setminus A' = \emptyset$ and, since Q has no edges loops, we have $\text{Arr}(A \cap A', A \cap A') = \emptyset$, hence for any $y \in \text{Rep}(\alpha')$ the map $F_{\rho, y}$ has the form (we use (15), (16) and that now $\alpha_j = 1$):

$$F_{\rho, y} : \text{Hom}(k, k^{\alpha'_j}) \rightarrow \prod_{a \in \text{Arr}(\{j\}, A' \setminus \{j\})} \text{Hom}(k, k^{\alpha'_{t(a)}})$$

$$F_{\rho, y}(f) = \{-y_a \circ f\}_{a \in \text{Arr}(\{j\}, A' \setminus \{j\})}$$

Obviously, we can choose $y \in \text{Rep}(\alpha')$ so that $F_{\rho,y}$ has maximal rank. Therefore by Corollary 3.5 $F_{\rho,\rho'}$ has maximal rank as well.

If $A' = \{j\}$, then by the already proved $F_{\rho'^\vee, \rho^\vee}^{Q^\vee}$ has a maximal rank. Now we apply Lemma 3.3, (e) and obtain that $F_{\rho,\rho'}$ has maximal rank. \square

An example of a quiver Q and exceptional representations ρ, ρ' , s. t. $F_{\rho,\rho'}^Q$ is not of maximal rank is as follows:

$$(17) \quad Q = \begin{array}{ccc} \circ & \longrightarrow & \circ \\ \uparrow & & \downarrow \\ \circ & \longrightarrow & \circ \end{array} \quad \rho = \begin{array}{ccc} k & \longrightarrow & 0 \\ \uparrow & & \downarrow \\ k & \longrightarrow & k \end{array} \quad \rho' = \begin{array}{ccc} 0 & \longrightarrow & k \\ \uparrow & & \downarrow \\ k & \longrightarrow & k \end{array}.$$

One easily computes $\text{hom}(\rho, \rho') = 1$, $\langle \underline{\dim}(\rho), \underline{\dim}(\rho') \rangle = 0$, and hence $\text{hom}^1(\rho, \rho') = 1$. Furthermore, ρ, ρ' are exceptional representations. Now from Lemma 3.3 (c) it follows that $F_{\rho,\rho'}$ is not of maximal rank. Comparing with Corollary 2.5 (b) we obtain

Lemma 3.7. *The categories of representations of the quivers $\begin{array}{ccc} \circ & \longrightarrow & \circ \\ \uparrow & & \downarrow \\ \circ & \longrightarrow & \circ \end{array}$, $\begin{array}{ccc} \circ & \longrightarrow & \circ \\ \uparrow & & \uparrow \\ \circ & \longrightarrow & \circ \end{array}$ are not equivalent.*

In the next section we restrict our considerations to a quiver Q without loops.

4. REMARKS ABOUT $F_{\rho,\rho'}$ IN QUIVERS WITHOUT LOOPS

Throughout this section Q is quiver without loops (i. e. the underlying graph $\Gamma(Q)$ is simply connected), in particular there is at most one edge between any two vertices of Q . Here we consider exceptional representations whose dimension vectors take values in $\{0, 1\}$. These exceptional representations are said to have thin dimension vector (Definition 4.2). The main result of this section is that, when the graph of Q has no loops, then for any two exceptional representations ρ, ρ' with thin dimension vectors the linear map $F_{\rho,\rho'}^Q$ has maximal rank. The last Lemma 4.6 of this section considers some cases in which $A \cap A'$ is a single element set and where ρ, ρ' are not restricted to be with thin dimension vectors.

For any subset $X \subset V(Q)$ we denote by Q_X the quiver with $V(Q_X) = X$ and $\text{Arr}(Q_X) = \text{Arr}(X, X)$. We denote by ρ, ρ' two representations of Q . We denote by $\alpha, \alpha' \in \mathbb{N}^{V(Q)}$ their dimension vectors, and by $A = \text{supp}(\alpha) \subset V(Q)$, $A' = \text{supp}(\alpha') \subset V(Q)$ the supports of α, α' . If $\text{Arr}(A \setminus A', A' \setminus A) \neq \emptyset$, then by the simply-connectivity of Q it follows that $A \cap A' = \emptyset$ and then $F_{\rho,\rho'}$ is trivially injective(see (15)). Thus, we see

Lemma 4.1. *If $\text{Arr}(A \setminus A', A' \setminus A) \neq \emptyset$, then $F_{\rho,\rho'}$ has maximal rank.*

From now on we assume that $\text{Arr}(A \setminus A', A' \setminus A) = \emptyset$, and then the last row in (16) can be erased, and we have a disjoint union:

$$(18) \quad \text{Arr}(A, A') = \text{Arr}(A \cap A', A \cap A') \cup \text{Arr}(A \cap A', A' \setminus A) \cup \text{Arr}(A \setminus A', A \cap A').$$

Now we consider exceptional representations ρ, ρ' whose dimension vectors contain only units and zeroes. More precisely:

Definition 4.2. *A vector $\alpha \in \mathbb{N}^{V(Q)}$ is said to be thin if for any $i \in A$ we have $\alpha_i = 1$, where $A = \text{supp}(\alpha) \subset V(Q)$ is the support of α .*

Remark 4.3. If $\rho \in \text{Rep}_k(Q)$ is an exceptional representation with a thin dimension vector (thin exceptional representation), then the sub-quiver Q_A must be connected and one can assume that $\forall a \in \text{Arr}(A, A) \quad \rho_a = \text{Id}_k$.

Lemma 4.4. Let ρ and ρ' be exceptional representations with thin dimension vectors. Then $F_{\rho, \rho'}$ has maximal rank.

Proof. Due to the given conditions we can write:

$$(19) \quad \langle \underline{\dim}(\rho), \underline{\dim}(\rho') \rangle = \#(A \cap A') - \#(\text{Arr}(A, A'))$$

$$\prod_{i \in A \cap A'} k \xrightarrow{F_{\rho, \rho'}} \prod_{a \in \text{Arr}(A, A')} k \quad F_{\rho, \rho'}(\{f_i\}_{i \in A \cap A'}) = \left\{ \begin{array}{ll} f_{t(a)} - f_{s(a)} & a \in \text{Arr}(A \cap A', A \cap A') \\ -f_{s(a)} & a \in \text{Arr}(A \cap A', A' \setminus A) \\ f_{t(a)} & a \in \text{Arr}(A \setminus A', A \cap A') \end{array} \right\}.$$

We can assume that $A \cap A' \neq \emptyset$. Since Q_A and $Q_{A'}$ are connected, it follows that $Q_{A \cap A'} = Q_A \cap Q_{A'}$ is connected. Since there are no loops in $\Gamma(Q)$, the graph of $Q_{A \cap A'}$ is simply connected, therefore $\#(A \cap A') = \#(\text{Arr}(A \cap A', A \cap A')) + 1$. Putting (18) and the latter equality in (19) we obtain

$$(20) \quad \langle \underline{\dim}(\rho), \underline{\dim}(\rho') \rangle = 1 - \#(\text{Arr}(A \cap A', A' \setminus A)) - \#(\text{Arr}(A \setminus A', A \cap A')).$$

The following lemma will be helpful for the rest of the proof

Lemma 4.5. Let T be a quiver, s. t. $\Gamma(T)$ is simply-connected. Consider the linear map

$$(21) \quad F : \prod_{i \in V(T)} k \rightarrow \prod_{a \in \text{Arr}(T)} k \quad F(\{f_i\}_{i \in V(T)}) = \{f_{t(a)} - f_{s(a)}\}_{a \in \text{Arr}(T)}$$

For each $j \in V(T)$, each $x \in k$, and each $y \in \prod_{a \in \text{Arr}(T)} k$ there exists unique $\{f_i\}_{i \in V(T)} \in \prod_{i \in V(T)} k$ with $f_j = x$ and $F(\{f_i\}_{i \in V(T)}) = y$. In particular, for each $j \in V(T)$ the linear map

$$(22) \quad \prod_{i \in V(T)} k \rightarrow k \oplus \prod_{a \in \text{Arr}(T)} k \quad \{f_i\}_{i \in V(T)} \mapsto (f_j, \{f_{t(a)} - f_{s(a)}\}_{a \in \text{Arr}(T)})$$

is isomorphism.

Proof. Easy induction on the number of vertices. □

We will apply this lemma to $Q_{A \cap A'}$.

Consider first the case $\langle \underline{\dim}(\rho), \underline{\dim}(\rho') \rangle \geq 0$. We need to show that $F_{\rho, \rho'}$ is surjective. Now by (20) we have $1 \geq \text{Arr}(A \cap A', A' \setminus A) + \text{Arr}(A \setminus A', A \cap A')$, and then the map $F_{\rho, \rho'}$ is the same as one of the maps (21) or (22) corresponding to $T = Q_{A \cap A'}$, hence $F_{\rho, \rho'}$ is surjective.

In the case $\langle \underline{\dim}(\rho), \underline{\dim}(\rho') \rangle < 0$, we have $1 < \text{Arr}(A \cap A', A' \setminus A) + \text{Arr}(A \setminus A', A \cap A')$. Hence, for some projection π the map $\pi \circ F_{\rho, \rho'}$ is the same as the map (22) corresponding to $T = Q_{A \cap A'}$, hence $F_{\rho, \rho'}$ is injective. □

In the end we consider the map $F_{\rho, \rho'}$ in the case, when $A \cap A'$ has a single element.

Lemma 4.6. Let ρ, ρ' be exceptional representations, s. t. $A \cap A' = \{j\}$. Let $\Gamma(Q)$ does not split at j , i. e. the edges adjacent to j can be represented as follows $x \text{ --- } j \text{ --- } y$. Finally, assume that α is constant on $A \cap \{x, y, j\}$ or α' is constant on $A' \cap \{x, y, j\}$. Then $F_{\rho, \rho'}$ has maximal rank.

Proof. Since there are no loops in $\Gamma(Q)$, we have $\text{Arr}(A \cap A', A \cap A') = \emptyset$ and $F_{\rho, \rho'}$ has the form

$$(23) \quad \langle \underline{\dim}(\rho), \underline{\dim}(\rho') \rangle = \alpha_j \alpha'_j - \sum_{a \in \text{Arr}(\{j\}, A' \setminus A)} \alpha_j \alpha'_{t(a)} - \sum_{a \in \text{Arr}(A \setminus A', \{j\})} \alpha_{s(a)} \alpha'_j$$

$$(24) \quad F_{\rho, \rho'} : \text{Hom}(k^{\alpha_j}, k^{\alpha'_j}) \rightarrow \prod_{a \in \text{Arr}(\{j\}, A' \setminus A)} \text{Hom}(k^{\alpha_j}, k^{\alpha'_{t(a)}}) \oplus \prod_{a \in \text{Arr}(A \setminus A', \{j\})} \text{Hom}(k^{\alpha_{s(a)}}, k^{\alpha'_j})$$

$$(25) \quad F_{\rho, \rho'}(\{f\}) = \left\{ \begin{array}{ll} -\rho'_a \circ f & a \in \text{Arr}(\{j\}, A' \setminus A) \\ f \circ \rho_a & a \in \text{Arr}(A \setminus A', \{j\}) \end{array} \right\}.$$

From Lemma 3.6 we can assume that $\#(A) \geq 2$, $\#(A') \geq 2$. Since ρ, ρ' are exceptional representations, Q_A and $Q_{A'}$ are connected. Then the edges adjacent to j can be represented as follows $A \setminus A' \ni x \xrightarrow{\quad} j \xrightarrow{\quad} y \in A' \setminus A$. We consider three cases.

If $\text{Arr}(\{j\}, A' \setminus A) \neq \emptyset$, $\text{Arr}(A \setminus A', \{j\}) = \emptyset$, then we can represent the arrows adjacent to j as follows

$$(26) \quad A \setminus A' \ni x \xleftarrow{\quad} j \xrightarrow{a} y \in A' \setminus A$$

and $\langle \underline{\dim}(\rho), \underline{\dim}(\rho') \rangle = \alpha_j \alpha'_j - \alpha_j \alpha'_y = \alpha_j(\alpha'_j - \alpha'_y)$, $F_{\rho, \rho'}(\{f\}) = -\rho'_a \circ f$. Since ρ' is an exceptional representation, the map ρ'_a has maximal rank (see the last part of Corollary 3.5). Therefore $F_{\rho, \rho'}$ has maximal rank.

If $\text{Arr}(\{j\}, A' \setminus A) = \emptyset$, $\text{Arr}(A \setminus A', \{j\}) \neq \emptyset$, then we can represent the arrows adjacent to j as follows

$$(27) \quad A \setminus A' \ni x \xrightarrow{a} j \xleftarrow{\quad} y \in A' \setminus A$$

and $\langle \underline{\dim}(\rho), \underline{\dim}(\rho') \rangle = \alpha_j \alpha'_j - \alpha_x \alpha'_j = (\alpha_j - \alpha_x) \alpha'_j$, $F_{\rho, \rho'}(\{f\}) = f \circ \rho_a$. Since ρ is an exceptional representation, then ρ_a has maximal rank. Therefore $F_{\rho, \rho'}$ has maximal rank.

If $\text{Arr}(\{j\}, A' \setminus A) \neq \emptyset$, $\text{Arr}(A \setminus A', \{j\}) \neq \emptyset$, then we can represent the arrows adjacent to j as follows

$$(28) \quad A \setminus A' \ni x \xrightarrow{a} j \xrightarrow{b} y \in A' \setminus A$$

and $\langle \underline{\dim}(\rho), \underline{\dim}(\rho') \rangle = \alpha_j \alpha'_j - \alpha_x \alpha'_j - \alpha_j \alpha'_y$, $F_{\rho, \rho'}(\{f\}) = (-\rho'_b \circ f, f \circ \rho_a)$.

If α is constant on $A \cap \{x, y, j\}$, then $\alpha_x = \alpha_j$ and $\langle \underline{\dim}(\rho), \underline{\dim}(\rho') \rangle = -\alpha_j \alpha'_y < 0$. Since $\alpha_x = \alpha_j$ and ρ_a has maximal rank, it follows that ρ_a is isomorphism, hence $F_{\rho, \rho'}$ is injective. Therefore $F_{\rho, \rho'}$ has maximal rank.

If α' is constant on $A' \cap \{x, y, j\}$, then $\alpha'_y = \alpha'_j$ and $\langle \underline{\dim}(\rho), \underline{\dim}(\rho') \rangle = -\alpha'_j \alpha_x < 0$. Since $\alpha'_y = \alpha'_j$ and ρ'_b has maximal rank, it follows that ρ'_b is isomorphism, hence $F_{\rho, \rho'}$ is injective. Therefore $F_{\rho, \rho'}$ has maximal rank. The lemma is completely proved. \square

5. REMARKS ABOUT $F_{\rho, \rho'}$ IN STAR SHAPED QUIVERS

In this section 5 we restrict Q further. We assume that its graph is of the type $(m, n, p \geq 2)$:

$$(29) \quad \Gamma(Q) = u_1 \text{ --- } u_2 \cdots u_{m-1} \text{ --- } s \begin{array}{l} \nearrow v_{n-1} \\ \searrow w_{p-1} \end{array} \begin{array}{l} \nearrow v_1 \\ \nearrow v_2 \\ \vdots \\ \searrow w_2 \\ \searrow w_1 \end{array}$$

Everything in this section holds for star shaped quivers with more than three rays. For simplicity of the notations we work with three rays. For such quivers we consider exceptional representations with hill dimension vector (Definition 5.2) in addition to the already considered thin dimension vectors. We show that for any two exceptional representations $\rho, \rho' \in \text{Rep}_k(Q)$, whose dimension vectors are hill or thin, the map $F_{\rho, \rho'}^Q$ has maximal rank.

The following lemma (more precisely parts (a) and (b) in this lemma) is basic for this section.

Lemma 5.1. *Let L be a quiver whose vertices are the numbers $\{1, 2, \dots, n\}$ ($n \geq 2$), whose graph is $\Gamma(L) = 1 \text{ --- } 2 \text{ --- } \dots \text{ --- } (n-1) \text{ --- } n$, and with any orientation of the arrows. Let $\rho, \rho' \in \text{Rep}_k(L)$ be two representations with dimension vectors $\alpha = \underline{\dim}(\rho)$, $\alpha' = \underline{\dim}(\rho')$, s. t.*

$$0 < \alpha_1 \leq \alpha_2 \leq \dots \leq \alpha_{n-1} \leq \alpha_n \quad 0 < \alpha'_1 \leq \alpha'_2 \leq \dots \leq \alpha'_{n-1} \leq \alpha'_n,$$

and s. t. for each $a \in \text{Arr}(L)$ the linear maps ρ_a, ρ'_a have maximal rank. Then the linear map

$$(30) \quad \prod_{i=1}^n \text{Hom}(k^{\alpha_i}, k^{\alpha'_i}) \xrightarrow{F_{\rho, \rho'}^L} \prod_{a \in \text{Arr}(L)} \text{Hom}(k^{\alpha_{s(a)}}, k^{\alpha'_{t(a)}}) \quad \{f_i\}_{i=1}^n \mapsto \{f_{t(a)} \circ \rho_a - \rho'_a \circ f_{s(a)}\}_{a \in \text{Arr}(L)}$$

has the following properties:

- (a) The map $\ker(F_{\rho, \rho'}^L) \rightarrow \text{Hom}(k^{\alpha_n}, k^{\alpha'_n})$, defined by $\ker(F_{\rho, \rho'}^L) \ni \{f_i\}_{i=1}^n \mapsto f_n$, is injective.
- (b) For any $x \in \text{Hom}(k^{\alpha_1}, k^{\alpha'_1})$ and any $y \in \prod_{a \in \text{Arr}(L)} \text{Hom}(k^{\alpha_{s(a)}}, k^{\alpha'_{t(a)}})$ there exists $\{f_i\}_{i=1}^n \in \prod_{i=1}^n \text{Hom}(k^{\alpha_i}, k^{\alpha'_i})$, s. t. $F_{\rho, \rho'}^L(\{f_i\}_{i=1}^n) = y$ and $f_1 = x$.
- (c) In particular, the map $F_{\rho, \rho'}^L$ is surjective and $\dim(\ker(F_{\rho, \rho'}^L)) = \sum_{i=1}^n \alpha_i \alpha'_i - \sum_{a \in \text{Arr}(L)} \alpha_{s(a)} \alpha'_{t(a)}$.
- (d) If we are given a surjective map $k^{\alpha'_0} \xleftarrow{x} k^{\alpha'_1}$, then the linear map

$$(31) \quad \prod_{i=1}^n \text{Hom}(k^{\alpha_i}, k^{\alpha'_i}) \xrightarrow{G_{\rho, \rho', x}^L} \text{Hom}(k^{\alpha_1}, k^{\alpha'_0}) \oplus \prod_{a \in \text{Arr}(L)} \text{Hom}(k^{\alpha_{s(a)}}, k^{\alpha'_{t(a)}})$$

$$\{f_i\}_{i=1}^n \mapsto (-x \circ f_1, F_{\rho, \rho'}^L(\{f_i\}_{i=1}^n))$$

is surjective and the dimension of its kernel is $\sum_{i=1}^n \alpha_i \alpha'_i - \sum_{a \in \text{Arr}(L)} \alpha_{s(a)} \alpha'_{t(a)} - \alpha_1 \alpha'_0$.

(e) If we are given an injective map $k^{\alpha_0} \xrightarrow{x} k^{\alpha_1}$, then the linear map

$$(32) \quad \prod_{i=1}^n \text{Hom}(k^{\alpha_i}, k^{\alpha'_i}) \xrightarrow{H^L_{\rho_r, \rho'_r, x}} \text{Hom}(k^{\alpha_0}, k^{\alpha'_1}) \oplus \prod_{a \in \text{Arr}(L)} \text{Hom}(k^{\alpha_{s(a)}}, k^{\alpha'_{t(a)}})$$

$$\{f_i\}_{i=1}^n \mapsto (f_1 \circ x, F^L_{\rho, \rho'}(\{f_i\}_{i=1}^n))$$

is surjective and the dimension of its kernel is $\sum_{i=1}^n \alpha_i \alpha'_i - \sum_{a \in \text{Arr}(L)} \alpha_{s(a)} \alpha'_{t(a)} - \alpha_0 \alpha'_1$.

(f) If we are given an injective map $k^{\alpha'_n} \xrightarrow{x} k^{\alpha'_{n+1}}$, then the linear map

$$\prod_{i=1}^n \text{Hom}(k^{\alpha_i}, k^{\alpha'_i}) \longrightarrow \text{Hom}(k^{\alpha_n}, k^{\alpha'_{n+1}}) \oplus \prod_{a \in \text{Arr}(L)} \text{Hom}(k^{\alpha_{s(a)}}, k^{\alpha'_{t(a)}})$$

$$\{f_i\}_{i=1}^n \mapsto (-x \circ f_n, F^L_{\rho, \rho'}(\{f_i\}_{i=1}^n))$$

is injective.

Proof. We prove first (a) and (b). Let $n = 2$. We consider the two possible orientations of the arrow $1 \longrightarrow 2$.

If the arrow starts at 1, then consider the diagram

$$\begin{array}{ccc} k^{\alpha_1} & \xrightarrow{\rho} & k^{\alpha_2} \\ f_1 \downarrow & & f_2 \downarrow \\ k^{\alpha'_1} & \xrightarrow{\rho'} & k^{\alpha'_2} \end{array}$$

Now the map $F^L_{\rho, \rho'}$ is $F^L_{\rho, \rho'}(f_1, f_2) = f_2 \circ \rho - \rho' \circ f_1$ and ρ, ρ' are injective. If $F^L_{\rho, \rho'}(f_1, f_2) = 0$ and $f_2 = 0$, then $\rho' \circ f_1 = 0$, and by the injectivity of ρ' we obtain $f_1 = 0$. Thus, we obtain (a).

To show (b) we have to find $f_2 \in \text{Hom}(k^{\alpha_2}, k^{\alpha'_2})$, s. t. $f_2 \circ \rho - \rho' \circ x = y$ for any $x \in \text{Hom}(k^{\alpha_1}, k^{\alpha'_1})$ and any $y \in \text{Hom}(k^{\alpha_2}, k^{\alpha'_2})$. Since ρ is injective, then it has left inverse $\pi : k^{\alpha_2} \rightarrow k^{\alpha_1}$, and then we can choose $f_2 = (y + \rho' \circ x) \circ \pi$.

If the arrow starts at 2, then consider the diagram

$$\begin{array}{ccc} k^{\alpha_1} & \xleftarrow{\rho} & k^{\alpha_2} \\ f_1 \downarrow & & f_2 \downarrow \\ k^{\alpha'_1} & \xleftarrow{\rho'} & k^{\alpha'_2} \end{array}$$

Now the map $F^L_{\rho, \rho'}$ is $F^L_{\rho, \rho'}(f_1, f_2) = f_1 \circ \rho - \rho' \circ f_2$ and ρ, ρ' are surjective. If $F^L_{\rho, \rho'}(f_1, f_2) = 0$ and $f_2 = 0$, then $f_1 \circ \rho = 0$, and by the surjectivity of ρ we obtain $f_1 = 0$. Thus, we obtain (a).

To show (b) we have to find $f_2 \in \text{Hom}(k^{\alpha_2}, k^{\alpha'_2})$, s. t. $x \circ \rho - \rho' \circ f_2 = y$ for any $x \in \text{Hom}(k^{\alpha_1}, k^{\alpha'_1})$ and any $y \in \text{Hom}(k^{\alpha_2}, k^{\alpha'_2})$. Since ρ' is surjective, then it has right inverse $in : k^{\alpha'_1} \rightarrow k^{\alpha'_2}$, and then we can choose $f_2 = in \circ (x \circ \rho - y)$.

So far, we proved the lemma, when $n = 2$. Now by using induction and the already proved case $n = 2$ one can easily prove (a), (b) for each $n \geq 2$. The statements in (c), (d), (e), and (f) follow from (a) and (b). \square

In Lemma 5.4 we allow one of the components ρ, ρ' to be of a type different from thin. More precisely:

Definition 5.2. Let Q be a star shaped quiver (as in Figure (29)). We say that $\alpha \in \mathbb{N}^{V(Q)}$ is a hill vector if

$$(33) \quad \begin{aligned} \alpha(u_1) &\leq \alpha(u_2) \leq \cdots \leq \alpha(u_{m-1}) \leq \alpha(s), & \alpha(u_{m-1}) &> 0 \\ \alpha(v_1) &\leq \alpha(v_2) \leq \cdots \leq \alpha(v_{n-1}) \leq \alpha(s), & \alpha(v_{n-1}) &> 0 \\ \alpha(w_1) &\leq \alpha(w_2) \leq \cdots \leq \alpha(w_{p-1}) \leq \alpha(s), & \alpha(w_{p-1}) &> 0. \end{aligned}$$

The non-vanishing condition for $\alpha(u_{m-1})$, $\alpha(v_{n-1})$, $\alpha(w_{p-1})$ in this definition simplifies our considerations, but we suspect that this condition can be relaxed.

In the proof of Lemma 5.4 and Lemma 5.5 we use the following simple observation:

Lemma 5.3. Let $Y \subset V$ be a vector subspace in a vector space V and $\dim(Y) = y$, $\dim(V) = n$. Let $\{x_1, \dots, x_m\}$ be integers in $\{0, 1, \dots, n\}$.

(a) If $y + \sum_{i=1}^m x_i - mn \geq 0$, then one can choose vector subspaces $\{X_i \subset V\}_{i=1}^m$ so that $\dim(X_i) = x_i$ and

$$(34) \quad \dim\left(Y \cap \bigcap_{i=1}^m X_i\right) = y + \sum_{i=1}^m x_i - mn$$

(b) If $y + \sum_{i=1}^m x_i - mn < 0$, then one can choose vector subspaces $\{X_i \subset V\}_{i=1}^m$ so that $\dim(X_i) = x_i$ and

$$(35) \quad Y \cap \bigcap_{i=1}^m X_i = \{0\}.$$

Proof. (a) If $m = 1$, then we have $y + x_1 \geq n$. Therefore we can choose $X_1 \subset V$, so that $\dim(X_1) = x_1$ and $X_1 + Y = V$. Therefore by a well known formula, we have $n = \dim(X_1 + Y) = \dim(X_1) + \dim(Y) - \dim(X_1 \cap Y) = x_1 + y - \dim(X_1 \cap Y)$. Hence $\dim(X_1 \cap Y) = x_1 + y - n$.

Suppose that (a) holds for some $m \geq 1$ and take any collection of integers $\{x_1, \dots, x_m, x_{m+1}\}$ in $\{0, 1, \dots, n\}$, s. t. $y + \sum_{i=1}^{m+1} x_i - (m+1)n \geq 0$. We can rewrite the last inequality as follows $n \leq y + \sum_{i=1}^m x_i + x_{m+1} - mn$. On the other hand $x_{m+1} \leq n$, therefore

$$(36) \quad n \leq y + \sum_{i=1}^m x_i + x_{m+1} - mn \leq y + \sum_{i=1}^m x_i + n - mn \Rightarrow 0 \leq y + \sum_{i=1}^m x_i - mn.$$

Now by the induction assumption we obtain vector subspaces $\{X_i \subset V\}_{i=1}^m$ with $\{\dim(X_i) = x_i\}_{i=1}^m$ and $\dim(Y \cap \bigcap_{i=1}^m X_i) = y + \sum_{i=1}^m x_i - mn$. Now we have $\dim(Y \cap \bigcap_{i=1}^m X_i) + x_{m+1} \geq n$ and as in the case $m = 1$ we find a vector subspace $X_{m+1} \subset V$ with $\dim(X_{m+1}) = x_{m+1}$ and $\dim(Y \cap \bigcap_{i=1}^{m+1} X_i) = \dim(Y \cap \bigcap_{i=1}^m X_i) + x_{m+1} - n = y + \sum_{i=1}^{m+1} x_i - (m+1)n$. Thus, we proved (a).

(b) If $m = 1$, then the statement is obvious. Now we assume that we have (b) for some $m \geq 1$. Let $\{x_1, \dots, x_m, x_{m+1}\}$ be any collection of integers in $\{0, 1, \dots, n\}$, s. t. $y + \sum_{i=1}^{m+1} x_i - (m+1)n < 0$. If $y + \sum_{i=1}^m x_i - mn < 0$, then we use the induction assumption. If $y + \sum_{i=1}^m x_i - mn \geq 0$, then we use (a) to obtain vector subspaces $\{X_i \subset V\}_{i=1}^m$ with $\{\dim(X_i) = x_i\}_{i=1}^m$ and $\dim(Y \cap \bigcap_{i=1}^m X_i) = y + \sum_{i=1}^m x_i - mn$. Now we have $\dim(Y \cap \bigcap_{i=1}^m X_i) + x_{m+1} = y + \sum_{i=1}^{m+1} x_i - mn < n$, therefore we can choose $X_{m+1} \subset V$ with $\dim(X_{m+1}) = x_{m+1}$ and $Y \cap \bigcap_{i=1}^{m+1} X_i = \{0\}$. The lemma is proved. \square

Lemma 5.4. *Let $\rho, \rho' \in \text{Rep}_k(Q)$ be two exceptional representations with thin and hill dimension vectors, respectively. Then $F_{\rho, \rho'}$ and $F_{\rho', \rho}$ have maximal rank.*

Proof. We show first that $F_{\rho, \rho'}$ has maximal rank. Due to Lemma 3.6 we can assume that $\#(A) \geq 2$, hence $A \cap A' \neq \{s\}$ (we are given also $\alpha'(u_{m-1}) > 0$, $\alpha'(v_{n-1}) > 0$, $\alpha'(w_{p-1}) > 0$). Due to Lemma 4.6 we can assume that $\#(A \cap A') \geq 2$. Lemma 4.1 considers the case $\text{Arr}(A \setminus A', A' \setminus A) \neq \emptyset$, hence we can assume that $\text{Arr}(A \setminus A', A' \setminus A) = \emptyset$ and we can write

$$(37) \quad \langle \alpha, \alpha' \rangle = \sum_{i \in A \cap A'} \alpha'_i - \sum_{a \in \text{Arr}(A \cap A', A \cap A')} \alpha'_{t(a)} - \sum_{a \in \text{Arr}(A \cap A', A' \setminus A)} \alpha'_{t(a)} - \sum_{a \in \text{Arr}(A \setminus A', A \cap A')} \alpha'_{t(a)}$$

$$(38) \quad F_{\rho, \rho'} : \prod_{i \in A \cap A'} \text{Hom}(k, k^{\alpha'_i}) \rightarrow \prod_{a \in \text{Arr}(A, A')} \text{Hom}(k, k^{\alpha'_{t(a)}})$$

$$(39) \quad F_{\rho, \rho'}(\{f_i\}_{i \in A \cap A'}) = \left\{ \begin{array}{ll} f_{t(a)} - \rho'_a \circ f_{s(a)} & a \in \text{Arr}(A \cap A', A \cap A') \\ -\rho'_a \circ f_{s(a)} & a \in \text{Arr}(A \cap A', A' \setminus A) \\ f_{t(a)} & a \in \text{Arr}(A \setminus A', A \cap A') \end{array} \right\}.$$

We consider first the case (see (29)) $A \cap A' \subset \{u_1, u_2, \dots, u_{m-1}\}$. Now $\Gamma(Q_{A \cap A'})$ has the form $u_i \text{ --- } u_{i+1} \text{ --- } \dots \text{ --- } u_{i+k-1} \text{ --- } u_{i+k}$. Let us denote by ρ_r, ρ'_r the representations ρ, ρ' restricted to $Q_{A \cap A'}$. Then $Q_{A \cap A'}, \rho_r, \rho'_r$ satisfy the conditions in Lemma 5.1 (recall also the last statement in Corollary 3.5). Let us denote by a the arrow adjacent to $Q_{A \cap A'}$ at u_{i+k} . If a starts at u_{i+k} , i. e. it points towards the splitting point s , then, due to (33), $a \in \text{Arr}(A \cap A', A' \setminus A)$ and ρ'_a is injective. In this case $\pi \circ F_{\rho, \rho'}$, where π is some projection, is the same as the linear map in Lemma 5.1 (f) with $x = \rho'_a$, hence $F_{\rho, \rho'}$ is injective. Let the arrow a ends at u_{i+k} , then it is neither in $\text{Arr}(A \cap A', A' \setminus A)$ nor in $\text{Arr}(A \setminus A', A' \cap A)$. Let us denote by b the arrow adjacent to $Q_{A \cap A'}$ at u_i . Now if b starts at u_i and $u_{i-1} \in A'$, then $b \in \text{Arr}(A \cap A', A' \setminus A)$ and $F_{\rho, \rho'}$ is the same as the linear map in Lemma 5.1 (d) with $x = \rho'_b$. If b starts at u_i and $u_{i-1} \notin A'$, then $\text{Arr}(A \cap A', A' \setminus A) = \text{Arr}(A \setminus A', A' \cap A) = \emptyset$ and $F_{\rho, \rho'}$ is the same as $F_{\rho_r, \rho'_r}^{Q_{A \cap A'}}$ from Lemma 5.1. If b ends at u_i and $u_{i-1} \in A$, then $b \in \text{Arr}(A \setminus A', A' \cap A)$ and $F_{\rho, \rho'}$ is the same as the linear map in Lemma 5.1 (e) with $x = \text{Id}_k$. If b ends at u_i and $u_{i-1} \notin A$, then $\text{Arr}(A \cap A', A' \setminus A) = \text{Arr}(A \setminus A', A' \cap A) = \emptyset$ and $F_{\rho, \rho'}$ is the same as $F_{\rho_r, \rho'_r}^{Q_{A \cap A'}}$ from Lemma 5.1. Thus, we see that $F_{\rho, \rho'}$ has maximal rank, when $A \cap A' \subset \{u_1, u_2, \dots, u_{m-1}\}$.

Next, we consider the case $A \cap A' \subset \{u_1, u_2, \dots, u_{m-1}, s\}$ and $s \in A \cap A'$. Now $\Gamma(Q_{A \cap A'})$ has the form $u_i \text{ --- } u_{i+1} \text{ --- } \dots \text{ --- } u_{m-1} \text{ --- } s$ and v_{n-1}, w_{p-1} are not elements of A . We denote by ρ_r, ρ'_r the restrictions of ρ, ρ' to $Q_{A \cap A'}$. Let us denote the arrow between s and v_{n-1} by b and the arrow between s and w_{p-1} by c . If b and c both end at s , then $F_{\rho, \rho'}$ is one of the following three linear maps: $F_{\rho_r, \rho'_r}^{Q_{A \cap A'}}$ (see (30)), $G_{\rho_r, \rho'_r, x}^{Q_{A \cap A'}}$ (see (31)), $H_{\rho_r, \rho'_r, \text{Id}_k}^{Q_{A \cap A'}}$ (see (32)) considered in Lemma 5.1, hence $F_{\rho, \rho'}$ is surjective. It is useful to denote

$$(40) \quad S = \{b, c\} \cap \text{Arr}(A \cap A', A' \setminus A).$$

For $a \in S$ the linear map ρ'_a is surjective and

$$(41) \quad \forall a \in S \quad \dim(\ker(\rho'_a)) = \alpha'_s - \alpha'_{t(a)}$$

Looking at (39) we see that $F_{\rho, \rho'}$ has the form

$$(42) \quad F_{\rho, \rho'}(\{f_i\}_{i \in A \cap A'}) = \left(T(\{f_i\}_{i \in A \cap A'}), \{-\rho'_a \circ f_s\}_{a \in S} \right),$$

where T is one of $F_{\rho_r, \rho'_r}^{Q_{A \cap A'}}$, $G_{\rho_r, \rho'_r, x}^{Q_{A \cap A'}}$, $H_{\rho_r, \rho'_r, \text{Id}_k}^{Q_{A \cap A'}}$. In the tree cases $\mathbf{ktr}(T) \subset \mathbf{ktr}(F_{\rho_r, \rho'_r}^{Q_{A \cap A'}})$ and by Lemma 5.1 (a) the linear map $\mathbf{ktr}(T) \xrightarrow{\kappa} \text{Hom}(k, k^{\alpha_s})$ defined by projecting to the $\text{Hom}(k, k^{\alpha_s})$ -component is injective (here and in (42) the notation s is the splitting vertex in figure (29)). Now from (42) we see that

$$(43) \quad \dim(\mathbf{ktr}(F_{\rho, \rho'})) = \dim \left(\kappa(\mathbf{ktr} T) \cap \bigcap_{a \in S} \mathbf{ktr}(\rho'_a) \right)$$

On the other hand by and (c), (d), (e) in Lemma 5.1, (41), and the formula (37) one easily shows that

$$(44) \quad \langle \alpha, \alpha' \rangle = \langle \underline{\dim}(\rho), \underline{\dim}(\rho') \rangle = \dim(\kappa(\mathbf{ktr} T)) + \sum_{a \in S} \dim(\mathbf{ktr}(\rho'_a)) - \#(S) \alpha'_s.$$

The feature of ρ' , due to the fact that it is an exceptional representation, used so far is that ρ'_a is of maximal rank for any $a \in \text{Arr}(Q)$. All considerations hold for any $\tilde{\rho}' \in \text{Rep}(\alpha')$ s. t. $\tilde{\rho}'_a$ is of maximal rank for $a \in \text{Arr}(Q)$. For any such $\tilde{\rho}'$ we have $\kappa(\mathbf{ktr}(\tilde{T})) \subset k^{\alpha'_s}$, $\mathbf{ktr}(\tilde{\rho}'_a) \subset k^{\alpha'_s}$ for $a \in S$ and (43), (44) hold. If $\langle \alpha, \alpha' \rangle \geq 0$, then by Lemma 5.3 (a) we can choose $\{\tilde{\rho}'_a\}_{a \in S}$ (without changing the rest elements of ρ') so that

$$(45) \quad \langle \alpha, \alpha' \rangle = \dim(\kappa(\mathbf{ktr} T)) + \sum_{a \in S} \dim(\mathbf{ktr}(\tilde{\rho}'_a)) - \#(S) \alpha'_s = \dim \left(\kappa(\mathbf{ktr} T) \cap \bigcap_{a \in S} \mathbf{ktr}(\tilde{\rho}'_a) \right).$$

Therefore, by (43) we get $\dim(\mathbf{ktr}(F_{\rho, \tilde{\rho}'})) = \langle \underline{\dim}(\rho), \underline{\dim}(\tilde{\rho}') \rangle$, which implies that $F_{\rho, \tilde{\rho}'}$ is surjective (see Lemma 3.3 (b)). Now Corollary 3.5 shows that $F_{\rho, \rho'}$ has maximal rank. If $\langle \alpha, \alpha' \rangle < 0$, then by Lemma 5.3 (b) we can choose $\{\tilde{\rho}'_a\}_{a \in S}$ (without changing the rest elements of ρ') so that $\{0\} = \kappa(\mathbf{ktr} \tilde{T}) \cap \bigcap_{a \in S} \mathbf{ktr}(\tilde{\rho}'_a)$. Hence (43) implies that $F_{\rho, \tilde{\rho}'}$ is injective. Now Corollary 3.5 shows that $F_{\rho, \rho'}$ has maximal rank.

Finally, we consider the case $\{u_{m-1}, s\} \subset A \cap A' \not\subset \{u_1, u_2, \dots, u_{m-1}, s\}$. Now $v_{n-1} \in A \cap A'$ or $w_{p-1} \in A \cap A'$. We will give details about the case when $v_{n-1} \in A \cap A'$ and $w_{p-1} \in A \cap A'$. The steps for the other cases are the same. Now the quiver $Q_{A \cap A'}$ has the form

$$(46) \quad Q_{A \cap A'} = \begin{array}{c} \begin{array}{c} v_j \\ \vdots \\ v_{n-1} \\ \swarrow \quad \searrow \\ u_i \cdots u_{m-1} \text{ --- } s \\ \swarrow \quad \searrow \\ w_{p-1} \\ \vdots \\ w_k \end{array} \end{array}$$

Let us denote $L_u = Q_{A \cap A' \cap \{u_1, u_2, \dots, u_{m-1}, s\}}$, and let ρ_u, ρ'_u be the restrictions of ρ, ρ' to L_u . Similarly we obtain L_v, ρ_v, ρ'_v and L_w, ρ_w, ρ'_w . Then we can apply Lemma 5.1 to L_i, ρ_i, ρ'_i for $i \in \{u, v, w\}$.

Furthermore, we can express $F_{\rho, \rho'}$ as follows:

$$(47) \quad F_{\rho, \rho'}(\{f_i\}_{i \in A \cap A'}) = (T_u(\{f_i\}_{i \in V(L_u)}), T_v(\{f_i\}_{i \in V(L_v)}), T_w(\{f_i\}_{i \in V(L_w)})),$$

where for $i \in \{u, v, w\}$ the linear map T_i is one of $F_{\rho_i, \rho'_i}^{L_i}$ (see (30)), $G_{\rho_i, \rho'_i, x_i}^{L_i}$ (see (31)), $H_{\rho_i, \rho'_i, \text{Id}_k}^{L_i}$ (see (32)). Using (c), (d), (e) in Lemma 5.1 and (37) one easily shows that

$$(48) \quad \langle \alpha, \alpha' \rangle = \dim(\ker(T_u)) + \dim(\ker(T_v)) + \dim(\ker(T_w)) - 2\alpha'_s.$$

By Lemma 5.1 (a) the linear map $\ker(T_i) \xrightarrow{\kappa_i} \text{Hom}(k, k^{\alpha_s})$ defined by projecting to the $\text{Hom}(k, k^{\alpha_s})$ -component is injective for $i \in \{u, v, w\}$. From (47) one easily shows that

$$(49) \quad \dim(\ker(F_{\rho, \rho'})) = \dim(\kappa_u(\ker T_u) \cap \kappa_v(\ker T_v) \cap \kappa_w(\ker T_w)).$$

The obtained formulas hold for any $\tilde{\rho}' \in \text{Rep}(\alpha')$ s. t. $\tilde{\rho}'_a$ is of maximal rank for $a \in \text{Arr}(Q)$ (we denote the corresponding linear maps be \tilde{T}_i). Due to (12) and having that \tilde{T}_i is $F_{\rho_i, \tilde{\rho}'_i}^{L_i}$ or $G_{\rho_i, \tilde{\rho}'_i, x_i}^{L_i}$ or $H_{\rho_i, \tilde{\rho}'_i, \text{Id}_k}^{L_i}$ we can move $\kappa_i(\ker(\tilde{T}_i))$ inside k^{α_s} by varying $\tilde{\rho}' \in \text{Rep}(\alpha')$. Therefore, if $\langle \alpha, \alpha' \rangle \geq 0$, using (48) and Lemma 5.3 (a), we can ensure

$$(50) \quad \dim(\kappa_u(\ker \tilde{T}_u) \cap \kappa_v(\ker \tilde{T}_v) \cap \kappa_w(\ker \tilde{T}_w)) = \langle \alpha, \alpha' \rangle.$$

Therefore $\dim(\ker(F_{\rho, \tilde{\rho}'})) = \langle \dim(\rho), \dim(\tilde{\rho}') \rangle$, which implies that $F_{\rho, \tilde{\rho}'}$ is surjective (see Lemma 3.3 (b)). Now Corollary 3.5 shows that $F_{\rho, \rho'}$ has maximal rank. If $\langle \alpha, \alpha' \rangle < 0$, then, due to Lemma 5.3 (b) and (48), we can vary $\tilde{\rho}'$ so that $\{0\} = \kappa_u(\ker \tilde{T}_u) \cap \kappa_v(\ker \tilde{T}_v) \cap \kappa_w(\ker \tilde{T}_w)$. Hence (49) implies that $F_{\rho, \tilde{\rho}'}$ is injective. Now Corollary 3.5 shows that $F_{\rho, \rho'}$ has maximal rank.

So far we proved that $F_{\rho, \rho'}^Q$ has a maximal rank. The quiver Q^\vee and the representations ρ^\vee, ρ'^\vee satisfy the same conditions as Q, ρ, ρ' , respectively. Therefore $F_{\rho^\vee, \rho'^\vee}^{Q^\vee}$ has maximal rank. Now Lemma 3.3 (e) shows that $F_{\rho', \rho}^Q$ has maximal rank. The lemma is proved. \square

Lemma 5.5. *Let $\rho, \rho' \in \text{Rep}_k(Q)$ be exceptional representations with hill dimension vectors. Then $F_{\rho, \rho'}$ has maximal rank.*

Proof. From Definition 5.2 we see that $Q_{A \cap A'}$ is in Figure (46). Now the arguments are the same as the arguments after Figure (46) in the proof of Lemma 5.4. In this case we use Lemma 5.1 in its full generality, when both representations have non-decreasing dimension vectors, so far we used it with one constant dimension vector and one non-decreasing dimension vector. \square

Combining Lemmas 5.5, 5.4 and 4.4 we obtain:

Proposition 5.6. *Let Q be a star shaped quiver. For any two exceptional representations $\rho, \rho' \in \text{Rep}_k(Q)$, whose dimension vectors are hill or thin, the linear map $F_{\rho, \rho'}$ has maximal rank. In particular, for any two such ρ, ρ' we have $\text{hom}(\rho, \rho') = 0$ or $\text{hom}^1(\rho, \rho') = 0$.*

In the end we discuss hill dimension vectors:

Lemma 5.7. *Let Q be a star-shaped quiver. The sum of any two hill vectors in $\mathbb{N}^{V(Q)}$ is hill.*

Let s be the splitting vertex of Q . Let $\delta \in \mathbb{N}^{V(Q)}$ be a hill vector with $\delta(s) \geq 3$ such that in each ray of the form $x_1 \cdots x_{k-1} \cdots s$ we have $\delta(x_i) = \frac{i}{k}\delta(s)$ for $i = 1, 2, \dots, k-1$.⁵ Then

⁵In particular $\frac{\delta(s)}{k}$ is a positive integer and $\delta(s) \geq 2$.

- (a) For each thin vector $\alpha \in \mathbb{N}^{V(Q)}$ the vectors $\delta \pm \alpha$ are hill.
- (b) For each hill vector $\alpha \in \mathbb{N}^{V(Q)}$, s. t. in each ray of the form $x_1 \text{ --- } x_2 \cdots x_{k-1} \text{ --- } s$ we have $\alpha(x_1) \leq \frac{\delta(s)}{k}$, $\{\alpha(x_{i+1}) - \alpha(x_i) \leq \frac{\delta(s)}{k}\}_{i=1}^{k-1}$ and $\alpha(x_{k-1}) < \frac{k-1}{k}\delta(s)$ the vector $\delta - \alpha$ is hill.

Proof. Recalling the definition of hill vectors (Definition 5.2), it is clear that the sum of two hill vectors is hill. Let $x_1 \text{ --- } x_2 \cdots x_{k-1} \text{ --- } s$ be any ray of Q ($k \geq 2$) and let us denote $x_k = s$.

(a) If α is thin, then by $\{\delta(x_{i+1}) - \delta(x_i) = \frac{\delta(s)}{k} \geq 1\}_{i=1}^{k-1}$ one easily shows that $\alpha + \delta$ and $\alpha - \delta$ are non-decreasing along the ray towards the splitting vertex. By $\delta(s) \geq 3$ and the given properties of δ it follows that $\delta(x_{k-1}) > 1$ (otherwise $k = 2$ and $1 = \delta(s)/2$, which is a contradiction) now it is clear that $\delta \pm \alpha$ are hill.

(b) By the given properties of α we can represent it as follows $\alpha(x_1) = a_1$, $\alpha(x_2) = a_1 + a_2, \dots$, $\alpha(x_s) = \sum_{l=1}^k a_l$, where $0 \leq a_l \leq \frac{\delta(s)}{k}$ for $l = 1, 2, \dots, k$. Therefore $\delta(x_i) - \alpha(x_i) = \sum_{l=1}^i \left(\frac{\delta(s)}{k} - a_l \right)$ for $i = 1, 2, \dots, k$, so we see that $\delta - \alpha$ is non-decreasing along the ray towards the splitting vertex. Now $\alpha(x_{k-1}) < \frac{k-1}{k}\delta(s)$ implies that $\delta - \alpha$ is a hill vector. \square

6. REMARKS ABOUT $F_{\rho, \rho'}$ IN DYNKIN AND IN SOME EXTENDED DYNKIN QUIVERS.

A natural question motivated by Proposition 5.6 is whether the dimension vectors of all exceptional representations in a given star shaped quiver are either hill or thin ?

Recalling that the dimension vectors of the exceptional representations in any quiver are real roots (see the paragraph after (3)) and that for Dynkin quivers all roots are real, finding the answer of this question for Dynkin quivers is easy:

Remark 6.1. *Tables with the dimension vectors of roots of Dynkin quivers can be found in Patrick Browne's webpage [5]. In these tables one verifies that if Q is a Dynkin quiver (i. e. the graph of Q is A_n with $n \geq 1$ or D_n with $n \geq 4$ or E_n with $n = 6, 7, 8$), then all the roots of Q are either thin or hill. For the sake of clarity we give an explanation with pictures below.*

If $\Gamma(Q) = A_n$, then all roots are thin.

The $n(n-1)$ roots of a quiver with graph D_n ($n \geq 4$) are the roots with thin dimension vectors together with the roots of the following type:

$$0 \text{ --- } 0 \cdots 0 \text{ --- } 1 \text{ --- } 1 \cdots 1 \text{ --- } 2 \text{ --- } 2 \text{ --- } \cdots 2 \begin{array}{l} \nearrow 1 \\ \searrow 1 \end{array}$$

there are k zeroes, l ones, and m twos in the horizontal ray, where $m > 0$, $l > 0$, $k + l + m = n - 2$, $k < n - 3$. Thus, we see that when $\Gamma(Q) = D_n$, then all the roots are thin or hill.

The 36 roots of a quiver with graph E_6 are the roots with thin dimension vectors together with the roots coming from both the D_5 subgraphs, and the following 6 roots

$$\begin{array}{ccccc} \begin{array}{c} 1 \\ | \\ 1 \text{ --- } 1 \text{ --- } 2 \text{ --- } 2 \text{ --- } 1 \end{array} & \begin{array}{c} 1 \\ | \\ 1 \text{ --- } 2 \text{ --- } 2 \text{ --- } 2 \text{ --- } 1 \end{array} & \begin{array}{c} 1 \\ | \\ 1 \text{ --- } 1 \text{ --- } 2 \text{ --- } 1 \text{ --- } 1 \end{array} \\ \begin{array}{c} 1 \\ | \\ 1 \text{ --- } 2 \text{ --- } 3 \text{ --- } 2 \text{ --- } 1 \end{array} & \begin{array}{c} 1 \\ | \\ 1 \text{ --- } 2 \text{ --- } 2 \text{ --- } 1 \text{ --- } 1 \end{array} & \begin{array}{c} 2 \\ | \\ 1 \text{ --- } 2 \text{ --- } 3 \text{ --- } 2 \text{ --- } 1 \end{array} \end{array}$$

Therefore all roots are thin or hill.

Note that E_7 is obtained from E_6 by adding one vertex. From any root of E_6 we can obtain a root of E_7 by inserting a number which is the same as an adjacent number. For example from an E_6 -root $\begin{smallmatrix} 2 \\ | \\ 1-2-3-2-1 \end{smallmatrix}$ we obtain E_7 -roots: $\begin{smallmatrix} 2 \\ | \\ 1-2-3-2-2-1 \end{smallmatrix}$; $\begin{smallmatrix} 2 \\ | \\ 1-2-3-3-2-1 \end{smallmatrix}$; $\begin{smallmatrix} 2 \\ | \\ 1-2-3-2-1-1 \end{smallmatrix}$. We say that these roots are obtained by inserting from the corresponding E_6 root. Note that a root obtained by inserting from a hill E_6 -root is a hill E_7 -root.

The 63 roots of a quiver with graph E_7 are the roots with thin dimension vectors, together with the roots coming from the D_6 , E_6 subgraphs, the roots obtained by inserting from the E_6 roots, and the following 3 roots

$$\begin{smallmatrix} 2 \\ | \\ 1-2-4-3-2-1 \end{smallmatrix} \quad \begin{smallmatrix} 2 \\ | \\ 1-3-4-3-2-1 \end{smallmatrix} \quad \begin{smallmatrix} 2 \\ | \\ 2-3-4-3-2-1 \end{smallmatrix}$$

which are also hill dimension vectors.

Finally, the 120 roots of a quiver with graph E_8 are the roots with thin dimension vectors, together with the roots coming from the D_7 , E_7 subgraphs, the roots obtained by inserting from the E_7 roots, and the following 11 roots

$$\begin{array}{cc} \begin{smallmatrix} 2 \\ | \\ 1-3-5-4-3-2-1 \end{smallmatrix} & \begin{smallmatrix} 2 \\ | \\ 2-3-5-4-3-2-1 \end{smallmatrix} \\ \begin{smallmatrix} 3 \\ | \\ 1-3-5-4-3-2-1 \end{smallmatrix} & \begin{smallmatrix} 3 \\ | \\ 2-3-5-4-3-2-1 \end{smallmatrix} \\ \begin{smallmatrix} 2 \\ | \\ 2-4-5-4-3-2-1 \end{smallmatrix} & \begin{smallmatrix} 3 \\ | \\ 2-4-5-4-3-2-1 \end{smallmatrix} \\ \begin{smallmatrix} 3 \\ | \\ 2-4-6-4-3-2-1 \end{smallmatrix} & \begin{smallmatrix} 3 \\ | \\ 2-4-6-5-3-2-1 \end{smallmatrix} \\ \begin{smallmatrix} 3 \\ | \\ 2-4-6-5-4-2-1 \end{smallmatrix} & \begin{smallmatrix} 3 \\ | \\ 2-4-6-5-4-3-1 \end{smallmatrix} \\ & \begin{smallmatrix} 3 \\ | \\ 2-4-6-5-4-3-2 \end{smallmatrix}, \end{array}$$

which are hill dimension vectors as well.

For those extended Dynkin diagrams, which are star-shaped with three rays, the answer is also positive (which is probably known):

Lemma 6.2. *All the real roots of the extended Dynkin diagrams \tilde{E}_6 , \tilde{E}_7 , \tilde{E}_8 (see for example [3, fig. (4.13)]) are either hill or thin.*

Proof. Let \tilde{Q} be some of the listed extended Dynkin diagrams. There is a vertex $\star \in V(\tilde{Q})$ in the end of one of the rays, called extended vertex, s. t. after removing it and the connecting edge we obtain a corresponding embedded Dynkin diagram, which we denote by Q . Using the inclusion $V(Q) \subset V(\tilde{Q})$, one can consider the roots of Q as a subset of the real roots of \tilde{Q} .

Let $(\alpha, \beta) = \frac{1}{2}(\langle \alpha, \beta \rangle_{\tilde{Q}} + \langle \beta, \alpha \rangle_{\tilde{Q}})$ be the symmetrization of $\langle, \rangle_{\tilde{Q}}$. Then by definition (see [9, p. 15, 17]) we have $\Delta^{re}(\tilde{Q}) = \{\alpha \in \mathbb{Z}^{V(\tilde{Q})} : (\alpha, \alpha) = 1\}$ and $\Delta(Q) = \{\alpha \in \Delta^{re}(\tilde{Q}) : \alpha(e) = 0\}$. Let $\delta \in \mathbb{N}_{\geq 1}^{V(\tilde{Q})}$ be the minimal imaginary root of $\Delta_+(\tilde{Q})$. In [3, fig. (4.13)] are given the coordinates of δ for all extended Dynkin diagrams, and one sees that the component of δ at \star is $\delta(\star) = 1$. Furthermore, δ is a hill dimension vector satisfying the conditions in Lemma 5.7 (applied to \tilde{Q}).

Let us take any $\alpha \in \Delta_+^{re}(\tilde{Q})$. We will show that α is hill or thin. We can assume that $\alpha \notin \mathbb{Z}\delta$. If $\alpha(e) = 0$, then $\alpha \in \Delta_+(Q)$ and the lemma follows from Remark 6.1, so we can assume that $\alpha(e) = a \geq 1$. Since $(\delta, -) = 0$, we see that $\alpha - a\delta \in \Delta(Q)$. It follows that either $\alpha = a\delta + x$ or $\alpha = a\delta - x$ for some $x \in \Delta_+(Q)$. If $\alpha = a\delta + x$, then we immediately see that α is hill by Lemma 5.7 (a) and Remark 6.1. In the case $\alpha = a\delta - x$ we write $\alpha = (a-1)\delta + (\delta - x)$. Using Lemma 5.7 and the explanation given in Remark 6.1, one can show that for any $x \in \Delta_+(Q)$ the vector $\delta - x$ is thin or hill (Lemma 5.7 is helpful in showing this, but also some case by case checks are necessary). Applying again Lemma 5.7 we see that $\alpha = (a-1)\delta + (\delta - x)$ is thin or hill and the lemma follows. \square

Due to Remark 6.1 and Lemma 6.2, Proposition 5.6 implies immediately

Corollary 6.3. *Let Q be a Dynkin quiver or an extended Dynkin quiver of type $\tilde{\mathbb{E}}_6, \tilde{\mathbb{E}}_7, \tilde{\mathbb{E}}_8$. Then for any two exceptional representations $\rho, \rho' \in \text{Rep}_k(Q)$ the linear map $F_{\rho, \rho'}$ has maximal rank.*

In particular, for any two exceptional representations $\rho, \rho' \in \text{Rep}_k(Q)$ we have $\text{hom}(\rho, \rho') = 0$ or $\text{hom}^1(\rho, \rho') = 0$.

The part of Corollary 6.3 concerning Dynkin quivers follows easily from the fact that $\text{Rep}_k(Q)$ is representation directed for Dynkin Q (see [21, p. 59] for the argument and [4], [12] for the fact that Dynkin quivers are representation directed).⁶

Corollary 6.4. *If Q is a Dynkin quiver, then there are no Ext-nontrivial couples in $\text{Rep}_k(Q)$, i. e. for any two exceptional representations $\rho, \rho' \in \text{Rep}_k(Q)$ we have $\text{hom}^1(\rho, \rho') = 0$ or $\text{hom}^1(\rho', \rho) = 0$.*

Proof. Recall that for such a quiver we have $\langle \alpha, \alpha \rangle > 0$ for each $\alpha \in \mathbb{N}^{V(Q)} \setminus \{0\}$. Since for any two exceptional representations $\rho, \rho' \in \text{Rep}_k(Q)$ we have $\text{hom}(\rho, \rho') = 0$ or $\text{hom}^1(\rho, \rho') = 0$, we can apply Lemma 3.1. The corollary follows. \square

Corollary 6.5. *If Q is a Dynkin quiver, then $\text{Rep}_k(Q)$ is regularity preserving category. Furthermore, there are no σ -irregular objects for any $\sigma \in \text{Stab}(D^b(\text{Rep}_k(Q)))$.*

Proof. Since there are no Ext-nontrivial couples, RP properties 1, 2 (Definition 2.2) are tautologically satisfied. Then by [10, Proposition 6.6] $\text{Rep}_k(Q)$ is regularity preserving. Actually, due to [10, Lemma 6.3], there are no σ -irregular objects for any $\sigma \in \text{Stab}(D^b(\text{Rep}_k(Q)))$. \square

7. SOME DIRECTIONS OF FUTURE RESEARCH

Motivated by Corollaries 6.3, 6.4, 6.5 and the results in [10] we conjecture

Conjecture 7.1. *For each Dynkin quiver Q and each $\sigma \in \text{Stab}(D^b(Q))$ there exists a full σ -exceptional collection.*

⁶We thank Pranav Pandit for pointing us this fact.

When there are Ext-nontrivial couples, RP property 1 and RP property 2 are our method to prove regularity-preserving. The fact that they hold not only in $\text{Rep}_k(Q_1)$, but also in $\text{Rep}_k(Q_2)$ (Corollary 2.5) seems to be a trace of a larger unexplored picture. We expect that there are further non-trivial examples of regularity-preserving categories. For example, we conjecture

Conjecture 7.2. *For each quiver Q , whose underlying graph is a loop, there is a choice of the orientation of the arrows such that the category $\text{Rep}_k(Q)$ is regularity preserving.*

We showed in [10] that $\text{Rep}_k(Q_2)$ is regularity-preserving, but did not answer the question: is there a σ -exceptional quadruple for each $\sigma \in \text{Stab}(D^b(Q_2))$ (the Q_2 -analogue of Theorem 1.1). The results of [10, Sections 7,8, Subsection 9.1] hold for $\text{Rep}_k(Q_2)$. These are clues for a positive answer. In section [10, Section 2] are given the dimensions of $\text{Hom}(X, Y)$, $\text{Ext}^1(X, Y)$ for any two exceptional objects X, Y in Q_2 as well. This lays a ground for working on the Q_2 -analogue of Theorem 1.1.

Using the results in Section 2 and Theorem 1.1 we study recently further the topology of $\text{Stab}(D^b(\text{Rep}_k(Q_1)))$ and, in particular, we expect to prove

Conjecture 7.3. *$\text{Stab}(D^b(Q_1))$ is simply-connected.*

Question 7.4. *In [10] and in the present paper, the notion of regularity-preserving category was defined and applied to categories of homological dimension 1. To define and study a relevant notion for higher dimensional categories is another direction of future research.*

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